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THE LOGARITHMIC SOLUTIONS OF THE HYPERGEOMETRIC EQUATION

BY

N. E. NÖRLUND

København 1963
i kommission hos Ejnar Munksgaard

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Synopsis

Logarithmic solutions of the hypergeometric differential equation of the second order are considered at length. In the two first chapters linear and quadratic transformations of these solutions are given. Chapter III deals with RIEMANN'S P-function in the cases in which one or more of the exponent differences are integers, and the last chapter is devoted to the confluent hypergeometric differential equation.

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Introduction

§ 1. The Differential Equation

$$z(1-z)\frac{d^2 y}{dz^2}+[\gamma-(\alpha+\beta+1)z]\frac{dy}{dz}-\alpha\beta y=0 \tag{1}$$

has been discussed in detail since the memorable papers of GAUSS, RIEMANN and KUMMER on the subject. KUMMER obtained twenty-four solutions represented by hypergeometric series if $\gamma-1$, $\alpha-\beta$ and $\gamma-\alpha-\beta$ are not integers. It seems worth while to consider more closely than has hitherto been done, the cases in which one or more of these numbers are integers. Logarithmic solutions of the equation (1) have been considered already by GAUSS, GOURSAT, LINDELÖF and BARNES. From more recent time may be mentioned Vol. I of *Higher Transcendental Functions* compiled by the staff of the Bateman Manuscript Project under the direction of ARTHUR ERDÉLYI. The logarithmic solutions will be investigated at length on the following pages.

In Chapter I we shall consider linear transformations of the logarithmic and other exceptional solutions of Euler's hypergeometric differential equation (1). In Chapter II some examples of quadratic transformations are given. Chapter III deals with Riemann's P-function and the last chapter is devoted to the confluent hypergeometric differential equation.

If γ is neither zero nor a negative integer and if $|z|<1$ the hypergeometric function is defined by the series

$$F(\alpha,\beta,\gamma;z)=\sum_{\nu=0}^{\infty}\frac{(\alpha)_\nu(\beta)_\nu}{\nu!(\gamma)_\nu}z^\nu,$$

where

$$(\alpha)_\nu=\frac{\Gamma(\alpha+\nu)}{\Gamma(\alpha)}.$$

Thus

$$(\alpha)_\nu=\alpha(\alpha+1)(\alpha+2)\dots(\alpha+\nu-1),$$

$$(\alpha)_{-\nu}=\frac{1}{(\alpha-1)(\alpha-2)(\alpha-3)\dots(\alpha-\nu)},$$

when ν is a positive integer.

Chapter I

Euler's hypergeometric differential equation

§ 2. Using Frobenius' method we put

$$y = \sum_{\nu=0}^{\infty} c_\nu(\varrho)\, z^{\varrho+\nu}. \tag{2}$$

Substitution into the differential equation (1) yields the identity

$$\varrho(\varrho+\gamma-1)\, c_0\, z^{\varrho-1} + \sum_{\nu=0}^{\infty} [(\varrho+1+\nu)(\varrho+\gamma+\nu)\, c_{\nu+1}(\varrho) - (\varrho+\alpha+\nu)(\varrho+\beta+\nu)\, c_\nu(\varrho)]\, z^{\varrho+\nu} = 0.$$

If we determine the $c_\nu(\varrho)$'s such that

$$(\varrho+1+\nu)(\varrho+\gamma+\nu)\, c_{\nu+1}(\varrho) = (\varrho+\alpha+\nu)(\varrho+\beta+\nu)\, c_\nu(\varrho), \quad \nu = 0, 1, 2, \dots \tag{3}$$

the series (2) will be a solution of the non-homogeneous equation

$$z(1-z)\frac{d^2y}{dz^2} + [\gamma - (\alpha+\beta+1)z]\frac{dy}{dz} - \alpha\beta y = \varrho(\varrho+\gamma-1)\, c_0(\varrho)\, z^{\varrho-1}. \tag{4}$$

From (3) we get

$$c_\nu(\varrho) = \frac{(\varrho+\alpha)_\nu\,(\varrho+\beta)_\nu}{(\varrho+1)_\nu\,(\varrho+\gamma)_\nu}\, c_0(\varrho), \quad \nu = 1, 2, 3, \dots \tag{5}$$

By setting $\varrho = 0$ or $\varrho = 1-\gamma$ and taking $c_0 = 1$ we see that (1) has the solutions

$$F(\alpha, \beta, \gamma; z), \tag{6}$$

$$z^{1-\gamma} F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma; z). \tag{7}$$

If γ is nonintegral, both of these solutions are applicable and they are linearly independent. If $\gamma = 1$ the two solutions become identical and if γ tends to an integer different from 1, one of them becomes meaningless.

1°. We suppose, first, that γ is an integer < 1 and that one at least of the parameters α and β is equal to one of the numbers $0, -1, -2, \dots \gamma$. Setting $\varrho = 0$ the equations (3) leave c_0 and $c_{1-\gamma}$ indeterminate and (1) has the solution

$$c_0 \sum_{\nu=0}^{-\gamma} \frac{(\alpha)_\nu\,(\beta)_\nu}{\nu!\,(\gamma)_\nu}\, z^\nu + c_{1-\gamma}\, z^{1-\gamma} F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma; z)$$

containing the two arbitrary constants c_0 and $c_{1-\gamma}$. For brevity we put

$$f(\alpha, \beta, \gamma; z) = \sum_{\nu=0}^{-\gamma} \frac{(\alpha)_\nu\,(\beta)_\nu}{\nu!\,(\gamma)_\nu}\, z^\nu. \tag{8}$$

Besides (7) we have then the rational solution $f(\alpha, \beta, \gamma; z)$.

2°. Next, we suppose that γ is an integer >1 and that one at least of the parameters α and β is a positive integer $<\gamma$. Taking $\varrho = 1-\gamma$, the equations (3) leave c_0 and $c_{\gamma-1}$ indeterminate. It follows that (1) has the solution

$$c_0\, z^{1-\gamma} \sum_{\nu=0}^{\gamma-2} \frac{(\alpha-\gamma+1)_\nu\,(\beta-\gamma+1)_\nu}{\nu!\,(2-\gamma)_\nu}\, z^\nu + c_{\gamma-1}\, F(\alpha, \beta, \gamma; z),$$

where c_0 and $c_{\gamma-1}$ are arbitrary constants. Besides (6) we then have the rational solution $z^{1-\gamma} f(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma; z)$.

3°. Now it is assumed that γ is a positive integer and neither α nor β is one of the numbers $1, 2, \ldots \gamma-1$. Putting $\varrho = 1-\gamma$ we can give $c_{\gamma-1}$ any value, and we get again the solution (6). The two roots of the indical equation then give the same solution. To obtain a second solution we observe that if we take $c_{\gamma-1} = 1$, the right-hand side of (4) has a zero of second order at $\varrho = 1-\gamma$. The function

$$\frac{\partial}{\partial\varrho} \sum_{\nu=0}^{\infty} c_\nu(\varrho)\, z^{\varrho+\nu}$$

therefore will satisfy the equation (1) when $\varrho = 1-\gamma$.
It has been shown by Frobenius that if $|z|<1$, the series (2) converges uniformly with respect to ϱ in any finite domain. The second solution can therefore be represented by the series

$$\sum_{\nu=0}^{\infty} \left(c'_\nu(\varrho) + c_\nu(\varrho)\log z\right) z^{\varrho+\nu}. \tag{9}$$

We denote it by $G(\alpha, \beta, \gamma; z)$. Hence for $|z|<1$

$$\left.\begin{aligned} G(\alpha, \beta, \gamma; z) &= \sum_{\nu=1}^{\gamma-1} (-1)^{\nu-1}(\nu-1)!\, \frac{(\alpha)_{-\nu}(\beta)_{-\nu}}{(\gamma)_{-\nu}}\, z^{-\nu} \\ &+ \sum_{\nu=0}^{\infty} \frac{(\alpha)_\nu(\beta)_\nu}{\nu!\,(\gamma)_\nu}\left([\alpha, \beta, \gamma; \nu] + \log z\right) z^\nu, \end{aligned}\right\} \tag{10}$$

where for $\nu = 1, 2, 3, \ldots$

$$[\alpha, \beta, \gamma; \nu] = \sum_{r=0}^{\nu-1}\left(\frac{1}{\alpha+r} + \frac{1}{\beta+r} - \frac{1}{\gamma+r} - \frac{1}{1+r}\right) =$$

$$\Psi(\alpha+r) - \Psi(\alpha) + \Psi(\beta+r) - \Psi(\beta) - \Psi(\gamma+r) + \Psi(\gamma) - \Psi(1+r) + \Psi(1)$$

and

$$[\alpha, \beta, \gamma; 0] = 0.$$

Here $\sum_{\nu=1}^{\gamma-1}$ denotes zero if $\gamma = 1$.
The first term on the right-hand side of (10) may be written

$$(-1)^\gamma \frac{\Gamma(\gamma)\Gamma(\gamma-1)\Gamma(\alpha-\gamma+1)\Gamma(\beta-\gamma+1)}{\Gamma(\alpha)\Gamma(\beta)}\, z^{1-\gamma} \sum_{\nu=0}^{\gamma-2} \frac{(\alpha-\gamma+1)_\nu(\beta-\gamma+1)_\nu}{\nu!\,(2-\gamma)_\nu}\, z^\nu.$$

If α tends to zero (10) reduces to

$$G(0,\beta,\gamma;z) = -\sum_{\nu=1}^{\gamma-1}\frac{(\beta)_{-\nu}\,z^{-\nu}}{\nu\,(\gamma)_{-\nu}} + \sum_{\nu=1}^{\infty}\frac{(\beta)_{\nu}\,z^{\nu}}{\nu\,(\gamma)_{\nu}} + \log z,$$

and if α tends to a negative integer, say $-p$, we get

$$\left.\begin{aligned} G(-p,\beta,\gamma;z) = -\sum_{\nu=1}^{\gamma-1}\frac{p!\,(\beta)_{-\nu}}{(\nu)_{p+1}\,(\gamma)_{-\nu}}\,z^{-\nu} + \sum_{\nu=1}^{p}\frac{(-p)_{\nu}\,(\beta)_{\nu}}{\nu!\,(\gamma)_{\nu}}\,[-p,\beta,\gamma;\nu]\,z^{\nu} \\ +(-1)^{p}\,p!\sum_{\nu=p+1}^{\infty}\frac{(\beta)_{\nu}\,z^{\nu}}{\nu(\nu-1)\ldots(\nu-p)\,(\gamma)_{\nu}} + F(-p,\beta,\gamma;z)\log z. \end{aligned}\right\} \quad (11)$$

Another solution is obtained if c_0 is chosen in the following manner

$$c_0(\varrho) = \frac{\Gamma(\varrho+\alpha)\,\Gamma(\varrho+\beta)}{\Gamma(\varrho+1)\,\Gamma(\varrho+\gamma)}. \quad (12)$$

From (5) it now follows that

$$c_{\nu}(\varrho) = \frac{\Gamma(\varrho+\alpha+\nu)\,\Gamma(\varrho+\beta+\nu)}{\Gamma(\varrho+1+\nu)\,\Gamma(\varrho+\gamma+\nu)}.$$

If neither α nor β is an integer $<\gamma$, the right hand side of (4) has again a zero of the second order at $\varrho = 1-\gamma$ and (9) is a solution of (1). We denote this solution by $c_0(0)\,g(\alpha,\beta,\gamma;z)$, where

$$\left.\begin{aligned} g(\alpha,\beta,\gamma;z) = \sum_{\nu=1}^{\gamma-1}(-1)^{\nu-1}(\nu-1)!\,\frac{(\alpha)_{-\nu}\,(\beta)_{-\nu}}{(\gamma)_{-\nu}}\,z^{-\nu} + \\ \sum_{\nu=0}^{\infty}\frac{(\alpha)_{\nu}\,(\beta)_{\nu}}{\nu!\,(\gamma)_{\nu}}\,[\Psi(\alpha+\nu)+\Psi(\beta+\nu)-\Psi(\gamma+\nu)-\Psi(1+\nu)+\log z]\,z^{\nu}. \end{aligned}\right\} \quad (13)$$

If α or β is a non-positive integer, (13) becomes meaningless as $\Psi(\alpha)$ has poles at $\alpha = 0, -1, -2, \ldots$ To be able to give α and β all possible values it is convenient to consider two other solutions of (1). These are obtained in the same manner if for $c_0(\varrho)$ instead of (12) we take one of the following fractions

$$\frac{1}{\Gamma(1-\varrho-\alpha)\,\Gamma(1-\varrho-\beta)\,\Gamma(\varrho+1)\,\Gamma(\varrho+\gamma)} \quad \text{or} \quad \frac{\Gamma(\varrho+\beta)}{\Gamma(1-\varrho-\alpha)\,\Gamma(\varrho+1)\,\Gamma(\varrho+\gamma)}.$$

We thus get two solutions denoted g_0 and g_1 respectively and defined by

$$\left.\begin{aligned} g_0(\alpha,\beta,\gamma;z) = \sum_{\nu=1}^{\gamma-1}(-1)^{\nu-1}(\nu-1)!\,\frac{(\alpha)_{-\nu}\,(\beta)_{-\nu}}{(\gamma)_{-\nu}}\,z^{-\nu} + \\ \sum_{\nu=0}^{\infty}\frac{(\alpha)_{\nu}\,(\beta)_{\nu}}{\nu!\,(\gamma)_{\nu}}\,[\Psi(1-\alpha-\nu)+\Psi(1-\beta-\nu)-\Psi(\gamma+\nu)-\Psi(1+\nu)+\log z]\,z^{\nu}, \end{aligned}\right\} \quad (14)$$

$$\left.\begin{aligned} g_1(\alpha,\beta,\gamma;z) = \sum_{\nu=1}^{\gamma-1}(-1)^{\nu-1}(\nu-1)!\frac{(\alpha)_{-\nu}(\beta)_{-\nu}}{(\gamma)_{-\nu}}z^{-\nu}+ \\ \sum_{\nu=0}^{\infty}\frac{(\alpha)_\nu(\beta)_\nu}{\nu!\,(\gamma)_\nu}\left[\Psi(1-\alpha-\nu)+\Psi(\beta+\nu)-\Psi(\gamma+\nu)-\Psi(1+\nu)+\log(-z)\right]z^\nu, \end{aligned}\right\} \quad (15)$$

where in (13) and (14) $\log z$ is negative for $0<z<1$, whereas in (15) $\log(-z)$ is negative for $-1<z<0$. The series on the right of (13), (14) and (15) converges for $0<|z|<1$. For large positive values of ν we have by Stirling's formula the asymptotic expansion

$$\Psi(\alpha+\nu) = \log\nu + \frac{B_1(\alpha)}{\nu} - \frac{B_2(\alpha)}{2\nu^2} + \ldots,$$

where $B_1(\alpha)$, $B_2(\alpha)$, ... are the Bernoulli polynomials. Hence

$$\Psi(\alpha+\nu)+\Psi(\beta+\nu)-\Psi(\gamma+\nu)-\Psi(1+\nu) = \frac{\alpha+\beta-\gamma-1}{\nu}+O\left(\frac{1}{\nu^2}\right).$$

It follows that for $z=1$ the series (13) e. g. is convergent if $\Re(\gamma-\alpha-\beta)>-1$, or if $\gamma-\alpha-\beta=-1$.
In (14) we suppose that neither α nor β is a positive integer.
In (15) we suppose that α is not a positive integer and β not an integer $<\gamma$. (15) remains valid after a passage to the limit (cf. (11)) if α tends to a negative integer, as

$$\Psi(1-\alpha-\nu) = \Psi(1-\alpha) + \sum_{s=0}^{\nu-1}\frac{1}{\alpha+s}.$$

(14) remains valid if α or β tends to a negative integer or zero.

$G(\alpha,\beta,\gamma;z)$, $g(\alpha,\beta,\gamma;z)$ and $g_0(\alpha,\beta,\gamma;z)$ are symmetrical functions of the two first parameters α and β. But $g_1(\alpha,\beta,\gamma;z)$ and $g_1(\beta,\alpha,\gamma;z)$ are two different solutions of (1) unless $\alpha-\beta$ is an integer, in which case they are coincident. Any of these logarithmic solutions forms a fundamental system with $F(\alpha,\beta,\gamma;z)$.

4°. If γ is a non-positive integer and neither α nor β is one of the numbers $0,-1,-2,\ldots\gamma$, it is seen in the same manner that (1) has the solutions $z^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;z)$ and $z^{1-\gamma}G(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;z)$.

§ 3. If we make the substitution $z'=1-z$ in (1) it is transformed into a hypergeometric differential equation with parameters α, β and $\alpha+\beta-\gamma+1$. It follows that (1) has the linearly independent solutions

$$F(\alpha,\beta,\alpha+\beta-\gamma+1;1-z), \quad (16)$$

$$(1-z)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1;1-z), \quad (17)$$

provided that $\alpha+\beta-\gamma$ is nonintegral. If $\alpha+\beta-\gamma=0$, $F(\alpha,\beta,1;1-z)$ and $G(\alpha,\beta,1;1-z)$ are two independent solutions. If $\alpha+\beta-\gamma$ is a positive integer and one at least of the parameters α and β is a positive integer $\leq\alpha+\beta-\gamma$, then (1) has the solutions

(16) and $(1-z)^{\gamma-\alpha-\beta} f(\gamma-\alpha, \gamma-\beta, \gamma-\alpha-\beta+1; 1-z)$. If $\alpha+\beta-\gamma$ is a positive integer and neither α nor β is a positive integer $\leq \alpha+\beta-\gamma$, then (1) has the solutions (16) and $G(\alpha, \beta, \alpha+\beta-\gamma+1; 1-z)$.

If $\alpha+\beta-\gamma$ is a negative integer and one at least of the parameters α and β is equal to a non-positive integer $> \alpha+\beta-\gamma$, then (17) and $f(\alpha, \beta, \alpha+\beta-\gamma+1; 1-z)$ are linearly independent solutions of (1).

If $\alpha+\beta-\gamma$ is a negative integer and neither α nor β equals a non-positive integer $> \alpha+\beta-\gamma$, then (1) has the solutions (17) and $(1-z)^{\gamma-\alpha-\beta} G(\gamma-\alpha, \gamma-\beta, \gamma-\alpha-\beta+1; 1-z)$.

§ 4. If we make the substitutions $z = \frac{1}{z_1}$, $y = z^{-\alpha} y_1$ in (1), it is transformed into a hypergeometric differential equation with parameters $\alpha, \alpha-\gamma+1, \alpha-\beta+1$. It follows that (1) has the linearly independent solutions

$$z^{-\alpha} F\left(\alpha, \alpha-\gamma+1, \alpha-\beta+1; \frac{1}{z}\right), \tag{18}$$

$$z^{-\beta} F\left(\beta, \beta-\gamma+1, \beta-\alpha+1; \frac{1}{z}\right), \tag{19}$$

provided that $\alpha-\beta$ is nonintegral. If $\alpha = \beta$, they are coincident and we have the solutions (18) and $z^{-\alpha} G\left(\alpha, \alpha-\gamma+1, 1; \frac{1}{z}\right)$.

If $\alpha-\beta$ is a positive integer and one at least of the numbers α and $\alpha-\gamma+1$ is equal to a positive integer $\leq \alpha-\beta$, them (18) and $z^{-\beta} f\left(\beta, \beta-\gamma+1, \beta-\alpha+1; \frac{1}{z}\right)$ are linearly independent solutions of (1). If $\alpha-\beta$ is a positive integer and neither α nor $\alpha-\gamma+1$ equals a positive integer $\leq \alpha-\beta$, then $z^{-\alpha} G\left(\alpha, \alpha-\gamma+1, \alpha-\beta+1; \frac{1}{z}\right)$ and (18) are independent solutions of (1).

G can of course be replaced by g, g_0, or g_1, except in the cases where one or more of these functions become meaningless.

§ 5. From the expansions in powers of z it follows immediately that the solutions defined in § 2 are connected by the following linear relations

$$g(\alpha, \beta, \gamma; z) = G(\alpha, \beta, \gamma; z) + [\Psi(\alpha) + \Psi(\beta) - \Psi(\gamma) - \Psi(1)] F(\alpha, \beta, \gamma; z), \tag{20}$$

$$g_0(\alpha, \beta, \gamma; z) = G(\alpha, \beta, \gamma; z) + [\Psi(1-\alpha) + \Psi(1-\beta) - \Psi(\gamma) - \Psi(1)] F(\alpha, \beta, \gamma; z), \tag{21}$$

$$g_1(\alpha, \beta, \gamma; z) = G(\alpha, \beta, \gamma; z) + [\Psi(1-\alpha) + \Psi(\beta) - \Psi(\gamma) - \Psi(1) \mp \pi i] F(\alpha, \beta, \gamma; z). \tag{22}$$

As

$$\Psi(1-\alpha) = \Psi(\alpha) + \pi \cot \pi\alpha,$$

it follows that

$$g_0(\alpha, \beta, \gamma; z) - g(\alpha, \beta, \gamma; z) = \frac{\pi \sin \pi (\alpha + \beta)}{\sin \pi\alpha \sin \pi\beta} F(\alpha, \beta, \gamma; z), \tag{23}$$

$$g_1(\alpha, \beta, \gamma; z) - g(\alpha, \beta, \gamma; z) = \frac{\pi e^{\mp \pi i \alpha}}{\sin \pi\alpha} F(\alpha, \beta, \gamma; z), \tag{24}$$

$$g_0(\alpha, \beta, \gamma; z) - g_1(\alpha, \beta, \gamma; z) = \frac{\pi e^{\pm \pi i \beta}}{\sin \pi\beta} F(\alpha, \beta, \gamma; z). \tag{25}$$

In each case the upper or lower sign is taken according as $I(z)$ is positive or negative. $g(\alpha, \beta, \gamma; z)$ is a one-valued analytic function of z within the domain $|\arg z| < \pi$ with a cross-cut along the real axis from $-\infty$ to 0; $g_1(\alpha, \beta, \gamma; z)$ is one-valued within the domain $|\arg(-z)| < \pi$ with a cross-cut along the real axis from 0 to $+\infty$. But $g_0(\alpha, \beta, \gamma; z)$ needs a cross-cut from $-\infty$ to 0 and another from 1 to ∞ to make it one-valued.

It was pointed out by Gauss that

$$\frac{d^n}{dz^n} F(\alpha, \beta, \gamma; z) = \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} F(\alpha + n, \beta + n, \gamma + n; z).$$

Also

$$\frac{d^n}{dz^n} g(\alpha, \beta, \gamma; z) = \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} g(\alpha + n, \beta + n, \gamma + n; z),$$

$$\frac{d^n}{dz^n} [z^{\gamma-1} g(\alpha, \beta, \gamma; z)] = (-1)^n (1-\gamma)_n z^{\gamma-n-1} g(\alpha, \beta, \gamma - n; z),$$

$$\frac{d^n}{dz^n} [z^{\alpha+n-1} g(\alpha, \beta, \gamma; z)] = (\alpha)_n z^{\alpha-1} g(\alpha + n, \beta, \gamma; z),$$

as is obvious by differentiation of the power series.

The hypergeometric function $F(\alpha, \beta, \gamma; z)$ satisfies the following fundamental relations due to Euler and Gauss:

$$F(\alpha, \beta, \gamma; z) = (1-z)^{-\beta} F\left(\gamma - \alpha, \beta, \gamma; \frac{z}{z-1}\right), \tag{26}$$

$$F(\alpha, \beta, \gamma; z) = (1-z)^{-\alpha} F\left(\alpha, \gamma - \beta, \gamma; \frac{z}{z-1}\right), \tag{27}$$

$$F(\alpha, \beta, \gamma; z) = (1-z)^{\gamma-\alpha-\beta} F(\gamma - \alpha, \gamma - \beta, \gamma; z). \tag{28}$$

The logarithmic solutions satisfy similar relations, but not quite the same. By changing the dependent and the independent variable it is easily verified that if (1) has the solution $G(\alpha, \beta, \gamma; z)$, the function $(1-z)^{-\beta} G\left(\gamma - \alpha, \beta, \gamma; \frac{z}{z-1}\right)$ is also a solution, and the first term in the expansions in powers of z is the same for both solutions. We therefore have a linear relation of the form

$$(1-z)^{-\beta} G\left(\gamma-\alpha, \beta, \gamma; \frac{z}{z-1}\right) - G(\alpha, \beta, \gamma; z) = CF(\alpha, \beta, \gamma; z),$$

C being a constant. Expanding in powers of z, we get, if we equate the constant terms on both sides,

$$C = \mp \pi i + \sum_{s=1}^{\gamma-1} \frac{1}{s} \frac{(\gamma-1)(\gamma-2)\ldots(\gamma-s)}{(\alpha-1)(\alpha-2)\ldots(\alpha-s)}.$$

But it is known that

$$\Psi(1-\alpha) - \Psi(\gamma-\alpha) = \sum_{s=1}^{\infty} \frac{1}{s} \frac{(\gamma-1)(\gamma-2)\ldots(\gamma-s)}{(\alpha-1)(\alpha-2)\ldots(\alpha-s)},$$

if $\Re(\gamma-\alpha) > 0$. As in the actual case γ is a positive integer, the series reduces to the first $\gamma-1$ terms and we get

$$\begin{aligned} C &= \mp \pi i + \Psi(1-\alpha) - \Psi(\gamma-\alpha) \\ &= \mp \pi i + \sum_{s=1}^{\gamma-1} \frac{1}{\alpha-s}. \end{aligned}$$

We thus obtain the formula

$$G(\alpha, \beta, \gamma; z) = (1-z)^{-\beta} G\left(\gamma-\alpha, \beta, \gamma; \frac{z}{z-1}\right) + \left(\pm \pi i + \sum_{s=1}^{\gamma-1} \frac{1}{s-\alpha}\right) F(\alpha, \beta, \gamma; z) \quad (29)$$

where the upper or lower sign is taken according as $I(z) \gtrless 0$. As $G(\alpha, \beta, \gamma; z)$ is a symmetrical function of the two first parameters α and β, it follows that

$$G(\alpha, \beta, \gamma; z) = (1-z)^{-\alpha} G\left(\alpha, \gamma-\beta, \gamma; \frac{z}{z-1}\right) + \left(\pm \pi i + \sum_{s=1}^{\gamma-1} \frac{1}{s-\beta}\right) F(\alpha, \beta, \gamma; z). \quad (30)$$

Combining these two formulae we get

$$G(\alpha, \beta, \gamma; z) = (1-z)^{\gamma-\alpha-\beta} G(\gamma-\alpha, \gamma-\beta, \gamma; z) - \sum_{s=1}^{\gamma-1} \left(\frac{1}{\alpha-s} + \frac{1}{\beta-s}\right) F(\alpha, \beta, \gamma; z). \quad (31)$$

It will be remembered that $G(\alpha, \beta, \gamma; z)$ only exists when α and β are different from $1, 2, \ldots \gamma-1$. The constants figuring in (29), (30) and (31) therefore are always finite. If $\gamma = 1$, (31) reduces to

$$G(\alpha, \beta, 1; z) = (1-z)^{1-\alpha-\beta} G(1-\alpha, 1-\beta, 1; z).$$

From (29) and (20) it now follows that

$$g(\alpha, \beta, \gamma; z) = (1-z)^{-\beta} g\left(\gamma-\alpha, \beta, \gamma; \frac{z}{z-1}\right) + (\pm \pi i - \pi \cot \pi\alpha) F(\alpha, \beta, \gamma; z). \quad (32)$$

If we use (24), this formula reduces to

$$g_1(\alpha,\beta,\gamma;z)=(1-z)^{-\beta}g\left(\gamma-\alpha,\beta,\gamma;\frac{z}{z-1}\right). \tag{33}$$

In the same way we see that

$$g_1(\beta,\alpha,\gamma;z)=(1-z)^{-\alpha}g\left(\alpha,\gamma-\beta,\gamma;\frac{z}{z-1}\right), \tag{34}$$

$$g_1(\alpha,\beta,\gamma;z)=(1-z)^{-\alpha}g_0\left(\alpha,\gamma-\beta,\gamma;\frac{z}{z-1}\right), \tag{35}$$

$$g_1(\beta,\alpha,\gamma;z)=(1-z)^{-\beta}g_0\left(\gamma-\alpha,\beta,\gamma;\frac{z}{z-1}\right), \tag{36}$$

$$g(\alpha,\beta,\gamma;z)=(1-z)^{-\beta}g_1\left(\gamma-\alpha,\beta,\gamma;\frac{z}{z-1}\right), \tag{37}$$

$$g_0(\alpha,\beta,\gamma;z)=(1-z)^{-\alpha}g_1\left(\alpha,\gamma-\beta,\gamma;\frac{z}{z-1}\right). \tag{38}$$

Combining these formulae we obtain

$$g(\alpha,\beta,\gamma;z)=(1-z)^{\gamma-\alpha-\beta}g_0(\gamma-\alpha,\gamma-\beta,\gamma;z), \tag{39}$$

$$g_0(\alpha,\beta,\gamma;z)=(1-z)^{\gamma-\alpha-\beta}g(\gamma-\alpha,\gamma-\beta,\gamma;z), \tag{40}$$

$$g_1(\alpha,\beta,\gamma;z)=(1-z)^{\gamma-\alpha-\beta}g_1(\gamma-\beta,\gamma-\alpha,\gamma;z).$$

§ 6. We have seen that if γ is a non-positive integer and the parameter β equals one of the numbers $0, -1, -2, \ldots \gamma$, then (1) has the solution (8). Consequently $(1-z)^{-\alpha}f\left(\alpha,\gamma-\beta,\gamma;\frac{z}{z-1}\right)$ is also a solution and furthermore (7) is a solution. There is thus a linear relation of the form

$$(1-z)^{-\alpha}f\left(\alpha,\gamma-\beta,\gamma;\frac{z}{z-1}\right)-f(\alpha,\beta,\gamma;z)=Cz^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;z), \tag{41}$$

C being a constant. If we expand in powers of z and equate the coefficients of $z^{1-\gamma}$, we obtain

$$C=\frac{(\alpha)_{1-\gamma}}{(1-\gamma)!}\sum_{\nu=0}^{\beta-\gamma}\frac{(\gamma-\beta)_\nu(\gamma-1)_\nu}{\nu!\,(\gamma)_\nu}.$$

By Vandermonde's theorem this reduces to

$$C=(-1)^{\beta-\gamma}\frac{\Gamma(\beta-\gamma+1)\,\Gamma(1-\beta)}{\Gamma(1-\gamma)\,\Gamma(2-\gamma)}\alpha(\alpha+1)(\alpha+2)\ldots(\alpha-\gamma). \tag{42}$$

In the same manner it is seen that

$$(1-z)^{-\beta} f\left(\gamma-\alpha,\,\beta,\,\gamma;\,\frac{z}{z-1}\right) = f(\alpha,\,\beta,\,\gamma;\,z), \tag{43}$$

where, as before, β equals one of the numbers $0, -1, -2, \ldots \gamma$ and α has any value. Combining (41) and (43) we get

$$(1-z)^{\gamma-\alpha-\beta} f(\gamma-\alpha,\gamma-\beta,\gamma;z) - f(\alpha,\beta,\gamma;z) = Cz^{1-\gamma} F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;z), \tag{44}$$

C being the constant (42). The gamma functions figuring in (42) are positive integers. C vanishes if, and only if, α equals one of the numbers $0, -1, -2, \ldots \gamma$. It follows that

$$f(\alpha,\,\beta,\,\gamma;\,z) = (1-z)^{-\alpha} f\left(\alpha,\,\gamma-\beta,\,\gamma;\,\frac{z}{z-1}\right),$$

$$f(\alpha,\,\beta,\,\gamma;\,z) = (1-z)^{\gamma-\alpha-\beta} f(\gamma-\alpha,\,\gamma-\beta,\,\gamma;\,z),$$

provided that both α and β are non-positive integers $\geq \gamma$.

§ 7. If γ is not an integer and $\alpha+\beta-\gamma$ is not a negative integer, (1) has the solutions (6), (7), and (16). These three solutions are connected by the well-known linear relation

$$\left.\begin{aligned} &\Gamma(1-\gamma)\,F(\alpha,\beta,\gamma;z) + \frac{\Gamma(\alpha-\gamma+1)\,\Gamma(\beta-\gamma+1)\,\Gamma(\gamma-1)}{\Gamma(\alpha)\,\Gamma(\beta)}\, z^{1-\gamma} F(\alpha-\gamma+1\,,\,\beta-\gamma+1\,,\,2-\gamma;z) = \\ &\qquad \frac{\Gamma(\alpha-\gamma+1)\,\Gamma(\beta-\gamma+1)}{\Gamma(\alpha+\beta-\gamma+1)}\,F(\alpha,\,\beta,\,\alpha+\beta-\gamma+1;\,1-z). \end{aligned}\right\} \tag{45}$$

If γ tends to a positive integer and α or β are not integers $<\gamma$, the expression on the left tends to a limit, as shown by Gauss [11] in a similar case. Putting $\gamma = 1+p-\varepsilon$, where $p = 0, 1, 2, \ldots$, the left-hand side of (45) can be written

$$\frac{\Gamma(p-\varepsilon)}{\Gamma(\alpha)\,\Gamma(\beta)} \sum_{\nu=0}^{p-1} \frac{\Gamma(\alpha-p+\varepsilon+\nu)\,\Gamma(\beta-p+\varepsilon+\nu)}{\nu!\,(1-p+\varepsilon)_\nu}\, z^{\nu-p+\varepsilon} +$$

$$\frac{(-1)^p\pi}{\sin\pi\varepsilon}\,\frac{1}{\Gamma(\alpha)\,\Gamma(\beta)} \sum_{\nu=0}^{\infty} \left[\frac{\Gamma(\alpha+\nu)\,\Gamma(\beta+\nu)}{\nu!\,\Gamma(p-\varepsilon+\nu+1)} - \frac{\Gamma(\alpha+\varepsilon+\nu)\,\Gamma(\beta+\varepsilon+\nu)}{\Gamma(1+\varepsilon+\nu)\,\Gamma(p+\nu+1)}\, z^{\varepsilon}\right] z^{\nu},$$

where $\sum_0^{p-1}$ is to be interpreted as zero, when $p=0$. If $|z|<1$ the series converges uniformly with respect to ε, and it is easily seen that if $\varepsilon \to 0$, this expression tends to $\frac{(-1)^{p+1}}{p!}\, g(\alpha,\beta,p+1;z)$. It follows that if γ tends to a positive integer, the equation (45) takes the form

$$g(\alpha,\beta,\gamma;z) = (-1)^{\gamma}\,\frac{\Gamma(\gamma)\,\Gamma(\alpha-\gamma+1)\,\Gamma(\beta-\gamma+1)}{\Gamma(\alpha+\beta-\gamma+1)}\,F(\alpha,\,\beta,\,\alpha+\beta-\gamma+1;\,1-z), \tag{46}$$

where it is supposed that α and β are not integers $< \gamma$. The function $g(\alpha, \beta, \gamma; z)$ thus is regular at $z = 1$. The origin and infinity are the only singularities for the branch under consideration. If in (46) we replace α and β by $\gamma - \alpha$ and $\gamma - \beta$, we get from (40)

$$g_0(\alpha, \beta, \gamma; z) = (-1)^\gamma \frac{\Gamma(\gamma)\,\Gamma(1-\alpha)\,\Gamma(1-\beta)}{\Gamma(\gamma-\alpha-\beta+1)} (1-z)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma-\alpha-\beta+1; 1-z), \quad (47)$$

provided that α and β are not positive integers. The function $g_0(\alpha, \beta, \gamma; z)$ thus has a branch-point at $z = 1$ unless $\gamma - \alpha - \beta$ is an integer. If it is positive, $z = 1$ is a zero for g_0, but if $\gamma - \alpha - \beta$ is a negative integer, the first terms in the series on the right vanish and the right-hand side of (47) reduces to the right-hand side of (46). Conversely, if $\gamma - \alpha - \beta$ is a positive integer, the first terms in (46) vanish, and (46) takes the form of (47). In the actual case α and β cannot be integers, but $\alpha + \beta$ is an integer and therefore $g = g_0$ by virtue of (23).

We now turn to the function g_1. From (33) and (46) we have

$$g_1(\alpha, \beta, \gamma; z) = (-1)^\gamma \frac{\Gamma(\gamma)\,\Gamma(1-\alpha)\,\Gamma(\beta-\gamma+1)}{\Gamma(\beta-\alpha+1)} (1-z)^{-\beta} F\left(\gamma-\alpha, \beta, \beta-\alpha+1; \frac{1}{1-z}\right).$$

Now, by (26)

$$F\left(\gamma-\alpha, \beta, \beta-\alpha+1; \frac{1}{1-z}\right) = \left(\frac{-z}{1-z}\right)^{-\beta} F\left(\beta-\gamma+1, \beta, \beta-\alpha+1; \frac{1}{z}\right).$$

We thus obtain

$$g_1(\alpha, \beta, \gamma; z) = (-1)^\gamma \frac{\Gamma(\gamma)\,\Gamma(1-\alpha)\,\Gamma(\beta-\gamma+1)}{\Gamma(\beta-\alpha+1)} (-z)^{-\beta} F\left(\beta-\gamma+1, \beta, \beta-\alpha+1; \frac{1}{z}\right), \quad (48)$$

provided that α and $\gamma - \beta$ are not positive integers. If $\alpha - \beta$ is a positive integer, the first terms in the series on the right vanish and we get

$$g_1(\alpha, \beta, \gamma; z) = (-1)^\gamma \frac{\Gamma(\gamma)\,\Gamma(1-\beta)\,\Gamma(\alpha-\gamma+1)}{\Gamma(\alpha-\beta+1)} (-z)^{-\alpha} F\left(\alpha-\gamma+1, \alpha, \alpha-\beta+1; \frac{1}{z}\right). \quad (49)$$

§ 8. We shall now consider the behavior of the logarithmic solutions as $z \to 1$. We have already seen that $g(z)$ is regular at $z = 1$. From (46) we get

$$g(\alpha, \beta, \gamma; 1) = (-1)^\gamma \frac{\Gamma(\gamma)\,\Gamma(\alpha-\gamma+1)\,\Gamma(\beta-\gamma+1)}{\Gamma(\alpha+\beta-\gamma+1)}, \quad (50)$$

and from (47) it follows that

$$\lim_{z\to 1} (1-z)^{\alpha+\beta-\gamma} g_0(\alpha, \beta, \gamma; z) = (-1)^\gamma \frac{\Gamma(\gamma)\,\Gamma(1-\alpha)\,\Gamma(1-\beta)}{\Gamma(\gamma-\alpha-\beta+1)}. \quad (51)$$

Using Gauss's formula

$$F(\alpha, \beta, \gamma; 1) = \frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}, \quad \Re(\gamma-\alpha-\beta) > 0 \quad (52)$$

we get from (48)

$$g_1(\alpha, \beta, \gamma; 1) = e^{\pm \pi i (\beta - \gamma)} \frac{\Gamma(\gamma)\Gamma(\beta-\gamma+1)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)}, \tag{53}$$

provided that $\Re(\gamma-\alpha-\beta) > 0$. The upper or lower sign is taken according as z tends to a point on the upper or lower edge of the cut from 0 to ∞. From (21) we obtain

$$G(\alpha, \beta, \gamma; 1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} [\Psi(\gamma) + \Psi(1) - \Psi(1-\alpha) - \Psi(1-\beta)], \tag{54}$$

provided that α and β are not positive integers and $\Re(\gamma-\alpha-\beta) > 0$. We next consider the equation (28). If we use (52), it follows that

$$\lim_{z \to 1} (1-z)^{\alpha+\beta-\gamma} F(\alpha, \beta, \gamma; z) = \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}, \quad \Re(\alpha+\beta-\gamma) > 0. \tag{55}$$

In a similar way we get from (31) using (54) and (55)

$$\lim_{z \to 1} (1-z)^{\alpha+\beta-\gamma} G(\alpha, \beta, \gamma; z) = \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)} [\Psi(\gamma) + \Psi(1) - \Psi(\alpha) - \Psi(\beta)], \tag{56}$$

provided that $\Re(\alpha+\beta-\gamma) > 0$. Dealing with the case when $\gamma = \alpha + \beta$, we observe that putting $\gamma = 1$ in (46), we obtain

$$F(\alpha, \beta, \alpha+\beta; z) = \frac{-\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} g(\alpha, \beta, 1; 1-z).$$

Dividing both sides by $\log(1-z)$, we get

$$\lim_{z \to 1} \frac{F(\alpha, \beta, \alpha+\beta; z)}{\log(1-z)} = \frac{-\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}. \tag{57}$$

Now from (20) it follows that

$$g(\alpha, \beta, \alpha+\beta; z) = G(\alpha, \beta, \alpha+\beta; z) +$$
$$[\Psi(\alpha) + \Psi(\beta) - \Psi(\alpha+\beta) - \Psi(1)] F(\alpha, \beta, \alpha+\beta; z).$$

The function on the left is regular at $z = 1$. Dividing both sides by $\log(1-z)$ and using (57), we therefore obtain

$$\lim_{z \to 1} \frac{G(\alpha, \beta, \alpha+\beta; z)}{\log(1-z)} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} [\Psi(\alpha) + \Psi(\beta) - \Psi(\alpha+\beta) - \Psi(1)]. \tag{58}$$

For the existence of $G(\alpha, \beta, \gamma; z)$ it is necessary to assume that α and β are different from $1, 2, \ldots \gamma - 1$, but the formula (54) has been proved only when α and β are not positive integers. If α tends to an integer $\geq \gamma$, the first factor on the right vanishes and the second factor tends to infinity, but the product remains finite. If $\alpha = \gamma + n$, where $n = 0, 1, 2, \ldots$, we have by Gauss's formula (52) that $\lim_{z \to 1} F(\gamma + n, \beta, \gamma; z) = 0$, provided that $\Re(\beta) < -n$. From (20) it now follows that

$$G(\gamma + n, \beta, \gamma; 1) = g(\gamma + n, \beta, \gamma; 1), \quad \Re(\beta) < -n.$$

and using (50) we obtain

$$G(\gamma + n, \beta, \gamma; 1) = \frac{(-1)^{\gamma}\Gamma(\gamma)\, n!}{(\beta + n)(\beta + n - 1)\ldots(\beta - \gamma + 1)}, \tag{59}$$

provided that $\Re(\beta + n) < 0$. It is easily seen that the right-hand side of (59) is the limit of the right-hand side of (54), as $\alpha \to \gamma + n$.

It may also be useful to observe that from (13) we obtain

$$\lim_{z \to 0} z^{\gamma - 1} g(\alpha, \beta, \gamma; z) = \frac{(-1)^{\gamma}\Gamma(\gamma)\Gamma(\gamma - 1)}{(1 - \alpha)_{\gamma - 1}(1 - \beta)_{\gamma - 1}}, \tag{60}$$

provided that $\gamma > 1$. But if $\gamma = 1$, we have

$$\lim_{z \to 0} \frac{g(\alpha, \beta, 1; z)}{\log z} = 1. \tag{61}$$

For the sake of completeness we shall add that for the function $f(z)$ defined by (8), we get by Vandermonde's theorem

$$f(\alpha, \beta, \gamma; 1) = \frac{\Gamma(\alpha - \gamma + 1)\Gamma(\beta - \gamma + 1)}{\Gamma(1 - \gamma)\Gamma(\alpha + \beta - \gamma + 1)}.$$

When α is a negative integer $-n$, the theorem becomes

$$f(-n, \beta, \gamma; 1) = \frac{(\gamma - \beta)_n}{(\gamma)_n}.$$

These special values for $z = 1$ are useful when we want to calculate the constants figuring in the linear relations between the solutions of (1).

For example, if γ is a positive integer and $\gamma - \alpha - \beta$ is not an integer, we have the relation

$$G(\alpha, \beta, \gamma; z) = C_1 F(\alpha, \beta, \alpha + \beta - \gamma + 1; 1 - z) +$$
$$C_2 (1 - z)^{\gamma - \alpha - \beta} F(\gamma - \alpha, \gamma - \beta, \gamma - \alpha - \beta + 1; 1 - z),$$

provided that α and $\beta \neq 1, 2, \ldots \gamma - 1$. Using (28), (54) and (56), we get for the constants C_1 and C_2 the following values

$$C_1 = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}[\Psi(\gamma) + \Psi(1) - \Psi(1 - \alpha) - \Psi(1 - \beta)],$$

$$C_2 = \frac{\Gamma(\gamma)\Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha)\Gamma(\beta)}[\Psi(\gamma) + \Psi(1) - \Psi(\alpha) - \Psi(\beta)].$$

Integral Representations

§ 9. The function $g(\alpha, \beta, \gamma; z)$ was defined when γ is a positive integer and α or β are not integers $<\gamma$. It is easily seen that this function in several ways may be represented by integrals of the Barnes type [4], e. g. we have

$$g(\alpha, \beta, \gamma; z) = \frac{(-1)^{\gamma}}{2\pi i} \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \int_{-i\infty}^{+i\infty} z^{-t}\,\Gamma(t)\,\Gamma(t-\gamma+1)\,\Gamma(\alpha-t)\,\Gamma(\beta-t)\,dt, \quad (62)$$

where the path of integration is indented so that the poles of $\Gamma(t)\Gamma(t-\gamma+1)$ lie to the left and the poles of $\Gamma(\alpha-t)\Gamma(\beta-t)$ to the right of it. This integral converges for $|\arg z| < 2\pi$. Evaluating the integral as $2\pi i$ times the sum of the residues of the integrand at the simple poles $t = \gamma-1, \gamma-2, \ldots 1$ and the poles of the second order $t = 0, -1, -2, \ldots$, we get the series on the right of (13). Putting $z = 1$ in (62) and using Barnes' lemma [4, p. 155], we get the formula (50). Differentiating both sides of (62) with respect to z and putting $z = 1$, we obtain the formula (46). This formula is equivalent to the integral representation

$$\left.\begin{aligned} & g(\alpha, \beta, \gamma; z) = \\ & (-1)^{\gamma} \frac{\Gamma(\gamma)\Gamma(\alpha-\gamma+1)\Gamma(\beta-\gamma+1)}{\Gamma(\alpha)\Gamma(\beta)\,2\pi i} \int_{-i\infty}^{+i\infty} (z-1)^{-t} \frac{\Gamma(t)\Gamma(\alpha-t)\Gamma(\beta-t)}{\Gamma(\alpha+\beta-\gamma+1-t)}\,dt, \end{aligned}\right\} \quad (63)$$

in which $|\arg(z-1)| < \pi$ and the contour passes between the poles of $\Gamma(t)$ and the poles of $\Gamma(\alpha-t)\Gamma(\beta-t)$. Furthermore we have

$$\left.\begin{aligned} & g(\alpha, \beta, \gamma; z) = \\ & \frac{(-1)^{\gamma}\Gamma(\gamma)}{2\pi i} z^{1-\gamma} \int_{-i\infty}^{+i\infty} (z-1)^{\gamma-1-t} \frac{\Gamma(t-\gamma+1)\Gamma(\alpha-t)\Gamma(\beta-t)}{\Gamma(\alpha+\beta-t)}\,dt, \end{aligned}\right\} \quad (64)$$

and the last integral is also convergent for $|\arg(z-1)| < \pi$.

For the function g_0 we obtain [28]

$$\begin{aligned} & g_0(\alpha, \beta, \gamma; z) = \\ & \frac{(-1)^{\gamma}}{2\pi i} \Gamma(\gamma)\Gamma(1-\alpha)\Gamma(1-\beta) \int_{\gamma-i\infty}^{\gamma+i\infty} z^{-t} \frac{\Gamma(t)\Gamma(t-\gamma+1)\,dt}{\Gamma(t-\alpha+1)\Gamma(t-\beta+1)} \end{aligned}$$

provided that[1] $0 < z < 1$ and $\Re(\gamma-\alpha-\beta) > -1$. The path of integration is a straight line parallel to the imaginary axis and intersecting the real axis at the point γ. Evaluating the latter integral as $2\pi i$ times the sum of the residues of the integrand at the poles $t = \gamma-1, \gamma-2, \gamma-3, \ldots$ we get the series on the right-hand side of (14).

[1] The integral is vanishing for $z > 1$. It is divergent for negative and for complex values of z.

Similarly, the function $g_1(z)$ can be represented by the following integral

$$g_1(\alpha, \beta, \gamma; z) = \frac{(-1)^\gamma}{2\pi i} \frac{\Gamma(\gamma)\Gamma(1-\alpha)}{\Gamma(\beta)} \int_{\gamma - i\infty}^{\gamma + i\infty} (-z)^{-t} \frac{\Gamma(t)\Gamma(t-\gamma+1)\Gamma(\beta-t)}{\Gamma(t-\alpha+1)} dt,$$

provided that $|\arg(-z)| < \pi$. The contour is a line parallel to the imaginary axis, except that it is curved, if necessary, so that the increasing sequence of poles $\beta, \beta+1, \beta+2, \ldots$ lie to the right, and the decreasing sequence of poles $\gamma-1, \gamma-2, \gamma-3, \ldots$ to the left of the path of integration. Evaluating the integral as $2\pi i$ times the sum of the residues at the poles $\gamma-1, \gamma-2, \gamma-3, \ldots$ we get the series on the right-hand side of (15), provided that $|z| < 1$, but if we assume $|z| > 1$ and take minus $2\pi i$ times the sum of the residues at the poles $\beta, \beta+1, \beta+2, \ldots$ we get the series on the right of (48).

Continuation Formulae

§ 10. In the following tables m, n, p, and q denote non-negative integers. In case of an ambigous sign the upper or lower sign is to be taken according as $I(z)$ is positive or negative.

We consider two independent solutions in the neighbourhood of a singular point, and we shall write down the formulae giving the analytic continuation of these solutions into the neighbourhood of another singular point. Thus, in the first case below, where $\alpha+\beta-\gamma$ is a non-negative integer q and γ not an integer, we get the formulae (1) and (4) from (46) and (47) by changing γ into $\alpha+\beta-\gamma+1$ and z into $1-z$. The four other formulae in this table are easily verified, being simple relations between rational functions after a factor eventually has been removed. If α and β are not integers, we use the formulae (1) and (4). If α or β is a non-positive integer, we have (2) and (4). If α or β is equal to $1, 2, \ldots q$, we take (3) and (6). Finally, if α or β is equal to $q+1, q+2, q+3, \ldots$, (1) and (5) are to be used. It follows that the solutions $F(\alpha, \beta, \gamma; z)$ and $z^{1-\gamma} F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma; z)$ have a logarithmic singularity at $z = 1$, provided that α and β are not integers. But if α or β is a negative integer or an integer $> q$, one of these solutions is regular at $z = 1$, and the other is logarithmic. Finally, if α or β is equal to one of the numbers $1, 2, \ldots q$, both solutions have a pole of order q at $z = 1$.

Considering the sixth case, where γ and $\gamma-\alpha-\beta$ are positive integers, we see that the first formula is merely another form of (46), and the second is the same as (47). The third is a combination of the two first using (25). By permutation of α and β the fourth follows from the third. The three following formulae are obvious, being relations between rational functions.

In a similar way all formulae in the following tables can be proved.

1. $\alpha+\beta-\gamma=q$, γ not an integer.

$$\left.\begin{array}{l} F(\alpha,\beta,\gamma;z)=\dfrac{(-1)^{\alpha+\beta-\gamma+1}\,\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)\,\Gamma(\alpha+\beta-\gamma+1)}\,g(\alpha,\beta,\alpha+\beta-\gamma+1;1-z),\\ \qquad |\arg(1-z)|<\pi,\quad \alpha \text{ and } \beta\neq q,\,q-1,\,q-2,\dots \end{array}\right\} \quad (1)$$

$$\left.\begin{array}{l} F(\alpha,\beta,\gamma;z)=\dfrac{\Gamma(\alpha-\gamma+1)\,\Gamma(\beta-\gamma+1)}{\Gamma(1-\gamma)\,\Gamma(\alpha+\beta-\gamma+1)}\,F(\alpha,\beta,\alpha+\beta-\gamma+1;1-z),\\ \alpha \text{ or } \beta=0,\,-1,\,-2,\dots \end{array}\right\} \quad (2)$$

$$\left.\begin{array}{l} F(\alpha,\beta,\gamma;z)=\dfrac{\Gamma(\gamma)\,\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)}\,(1-z)^{\gamma-\alpha-\beta}f(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1;1-z),\\ \alpha \text{ or } \beta=1,\,2,\dots q. \end{array}\right\} \quad (3)$$

$$\left.\begin{array}{c} z^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;z)=\\ \dfrac{(-1)^{\alpha+\beta-\gamma+1}\,\Gamma(2-\gamma)}{\Gamma(1-\alpha)\,\Gamma(1-\beta)\,\Gamma(\alpha+\beta-\gamma+1)}\,g_0(\alpha,\beta,\alpha+\beta-\gamma+1;1-z),\\ |\arg z|<\pi,\quad |\arg(1-z)|<\pi,\quad \alpha \text{ and } \beta\neq 1,\,2,\,3,\dots \end{array}\right\} \quad (4)$$

$$\left.\begin{array}{c} z^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;z)=\\ \dfrac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\gamma-1)\,\Gamma(\alpha+\beta-\gamma+1)}\,F(\alpha,\beta,\alpha+\beta-\gamma+1;1-z),\\ |\arg z|<\pi,\quad \alpha \text{ or } \beta=q+1,\,q+2,\,q+3,\dots \end{array}\right\} \quad (5)$$

$$\left.\begin{array}{c} z^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;z)=\\ \dfrac{\Gamma(2-\gamma)\,\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha-\gamma+1)\,\Gamma(\beta-\gamma+1)}\,z^{1-\gamma}(1-z)^{\gamma-\alpha-\beta}f(1-\alpha,1-\beta,\gamma-\alpha-\beta+1;1-z),\\ \alpha \text{ or } \beta=1,\,2,\dots q. \end{array}\right\} \quad (6)$$

2. $\gamma-\alpha-\beta=q$, γ not an integer.

$$\left.\begin{array}{c} F(\alpha,\beta,\gamma;z)=\\ \dfrac{(-1)^{\gamma-\alpha-\beta+1}\,\Gamma(\gamma)\,(1-z)^{\gamma-\alpha-\beta}}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma-\alpha-\beta+1)}\,g(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1;1-z),\\ |\arg(1-z)|<\pi,\quad \alpha \text{ and } \beta\neq 0,\,-1,\,-2,\dots \end{array}\right\} \quad (1)$$

$$\left.\begin{array}{c} F(\alpha,\beta,\gamma;z)=\\ \dfrac{\Gamma(1-\alpha)\,\Gamma(1-\beta)}{\Gamma(1-\gamma)\,\Gamma(\gamma-\alpha-\beta+1)}\,(1-z)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1;1-z),\\ \alpha \text{ or } \beta=-q,\,-q-1,\,-q-2,\dots \end{array}\right\} \quad (2)$$

$$\left. \begin{array}{l} F(\alpha, \beta, \gamma; z) = \dfrac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)} f(\alpha, \beta, \alpha+\beta-\gamma+1; 1-z), \\ \alpha \text{ or } \beta = 0, -1, -2, \ldots 1-q. \end{array} \right\} \quad (3)$$

$$\left. \begin{array}{l} z^{1-\gamma} F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma; z) = \\ \dfrac{(-1)^{\gamma-\alpha-\beta+1}\,\Gamma(2-\gamma)\,(1-z)^{\gamma-\alpha-\beta}}{\Gamma(\alpha-\gamma+1)\,\Gamma(\beta-\gamma+1)\,\Gamma(\gamma-\alpha-\beta+1)} g_0(\gamma-\alpha, \gamma-\beta, \gamma-\alpha-\beta+1; 1-z), \\ |\arg z| < \pi, \quad |\arg(1-z)| < \pi, \quad \alpha \text{ and } \beta \neq 1-q, 2-q, 3-q, \ldots \end{array} \right\} \quad (4)$$

$$\left. \begin{array}{l} z^{1-\gamma} F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma; z) = \\ \dfrac{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}{\Gamma(\gamma-1)\,\Gamma(\gamma-\alpha-\beta+1)} (1-z)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma-\alpha-\beta+1; 1-z), \\ |\arg z| < \pi, \quad \alpha \text{ or } \beta = 1, 2, 3, \ldots \end{array} \right\} \quad (5)$$

$$\left. \begin{array}{l} z^{1-\gamma} F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma; z) = \\ \dfrac{\Gamma(2-\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(1-\alpha)\,\Gamma(1-\beta)} z^{1-\gamma} f(\alpha-\gamma+1, \beta-\gamma+1, \alpha+\beta-\gamma+1; 1-z), \\ \alpha \text{ or } \beta = 0, -1, \ldots 1-q. \end{array} \right\} \quad (6)$$

3. $\gamma = 1+p$, $\gamma-\alpha-\beta$ not an integer.

$$\left. \begin{array}{l} F(\alpha, \beta, \gamma; z) = \dfrac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)} F(\alpha, \beta, \alpha+\beta-\gamma+1; 1-z) + \\ \dfrac{\Gamma(\gamma)\,\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)} (1-z)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma-\alpha-\beta+1; 1-z), \\ |\arg(1-z)| < \pi, \end{array} \right\} \quad (1)$$

$$\left. \begin{array}{l} g(\alpha, \beta, \gamma; z) = \\ (-1)^{\gamma} \dfrac{\Gamma(\gamma)\,\Gamma(\alpha-\gamma+1)\,\Gamma(\beta-\gamma+1)}{\Gamma(\alpha+\beta-\gamma+1)} F(\alpha, \beta, \alpha+\beta-\gamma+1; 1-z), \\ |\arg z| < \pi, \quad \alpha \text{ and } \beta \neq p, p-1, p-2, \ldots \end{array} \right\} \quad (2)$$

$$\left. \begin{array}{l} g_0(\alpha, \beta, \gamma; z) = \\ (-1)^{\gamma} \dfrac{\Gamma(\gamma)\,\Gamma(1-\alpha)\,\Gamma(1-\beta)}{\Gamma(\gamma-\alpha-\beta+1)} (1-z)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma-\alpha-\beta+1; 1-z), \\ |\arg z| < \pi, \quad |\arg(1-z)| < \pi, \quad \alpha \text{ and } \beta \neq 1, 2, 3, \ldots \end{array} \right\} \quad (3)$$

$$\left.\begin{array}{c} F(\alpha,\beta,\gamma;z)=\dfrac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}\,F(\alpha,\beta,\alpha+\beta-\gamma+1;1-z),\\ \alpha \text{ or } \beta=0,-1,-2,\ldots \end{array}\right\}\quad(4)$$

$$\left.\begin{array}{c} F(\alpha,\beta,\gamma;z)=\\ \dfrac{\Gamma(\gamma)\,\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)}\,(1-z)^{\gamma-\alpha-\beta}\,F(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1;1-z),\\ |\arg(1-z)|<\pi,\quad \alpha \text{ or } \beta=p+1,p+2,p+3,\ldots \end{array}\right\}\quad(5)$$

$$\left.\begin{array}{c} z^{1-\gamma}f(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;z)=\\ \dfrac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\gamma-1)\,\Gamma(\alpha+\beta-\gamma+1)}\,F(\alpha,\beta,\alpha+\beta-\gamma+1;1-z),\\ \alpha \text{ or } \beta=1,2,\ldots p. \end{array}\right\}\quad(6)$$

$$\left.\begin{array}{c} z^{1-\gamma}(1-z)^{\gamma-\alpha-\beta}f(1-\alpha,1-\beta,2-\gamma;z)=\\ \dfrac{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)\,(1-z)^{\gamma-\alpha-\beta}}{\Gamma(\gamma-1)\,\Gamma(\gamma-\alpha-\beta+1)}\,F(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1;1-z),\\ \alpha \text{ or } \beta=1,2,\ldots p. \end{array}\right\}\quad(7)$$

4. $\gamma=1-p,\quad \gamma-\alpha-\beta$ not an integer.

$$\left.\begin{array}{c} z^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;z)=\\ \dfrac{\Gamma(2-\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(1-\alpha)\,\Gamma(1-\beta)}\,F(\alpha,\beta,\alpha+\beta-\gamma+1;1-z)+\\ \dfrac{\Gamma(2-\gamma)\,\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha-\gamma+1)\,\Gamma(\beta-\gamma+1)}\,(1-z)^{\gamma-\alpha-\beta}\,F(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1;1-z),\\ |\arg z|<\pi,\quad |\arg(1-z)|<\pi. \end{array}\right\}\quad(1)$$

$$\left.\begin{array}{c} z^{1-\gamma}g(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;z)=\\ (-1)^{\gamma}\dfrac{\Gamma(2-\gamma)\,\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta-\gamma+1)}\,F(\alpha,\beta,\alpha+\beta-\gamma+1;1-z),\\ |\arg z|<\pi,\quad \alpha \text{ and } \beta\neq 0,-1,-2,\ldots \end{array}\right\}\quad(2)$$

$$\left.\begin{array}{c} z^{1-\gamma}g_0(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;z)=\\ (-1)^{\gamma}\dfrac{\Gamma(2-\gamma)\,\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}{\Gamma(\gamma-\alpha-\beta+1)}\,(1-z)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1;1-z),\\ |\arg z|<\pi,\quad |\arg(1-z)|<\pi,\quad \alpha \text{ and } \beta\neq 1-p,2-p,3-p,\ldots \end{array}\right\}\quad(3)$$

$$\left.\begin{array}{c} z^{1-\gamma} F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma; z) = \\ \dfrac{\Gamma(2-\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(1-\alpha)\,\Gamma(1-\beta)} F(\alpha, \beta, \alpha+\beta-\gamma+1; 1-z), \\ |\arg z| < \pi, \quad \alpha \text{ or } \beta = -p, -p-1, -p-2, \ldots \end{array}\right\} \quad (4)$$

$$\left.\begin{array}{c} z^{1-\gamma} F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma; z) = \\ \dfrac{\Gamma(2-\gamma)\,\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha-\gamma+1)\,\Gamma(\beta-\gamma+1)} (1-z)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma-\alpha-\beta+1; 1-z), \\ |\arg z| < \pi, \quad |\arg(1-z)| < \pi, \quad \alpha \text{ or } \beta = 1, 2, 3, \ldots \end{array}\right\} \quad (5)$$

$$\left.\begin{array}{c} f(\alpha, \beta, \gamma; z) = \dfrac{\Gamma(\alpha-\gamma+1)\,\Gamma(\beta-\gamma+1)}{\Gamma(1-\gamma)\,\Gamma(\alpha+\beta-\gamma+1)} F(\alpha, \beta, \alpha+\beta-\gamma+1; 1-z), \\ \alpha \text{ or } \beta = 0, -1, \ldots 1-p. \end{array}\right\} \quad (6)$$

$$\left.\begin{array}{c} (1-z)^{\gamma-\alpha-\beta} f(\gamma-\alpha, \gamma-\beta, \gamma; z) = \\ \dfrac{\Gamma(1-\alpha)\,\Gamma(1-\beta)}{\Gamma(1-\gamma)\,\Gamma(\gamma-\alpha-\beta+1)} (1-z)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma-\alpha-\beta+1; 1-z), \\ \alpha \text{ or } \beta = 0, -1, \ldots 1-p. \end{array}\right\} \quad (7)$$

5. $\gamma = 1+p, \quad \alpha+\beta-\gamma = q.$

$$\left.\begin{array}{c} F(\alpha, \beta, \gamma; z) = \\ \dfrac{(-1)^{\alpha+\beta-\gamma+1}\,\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)\,\Gamma(\alpha+\beta-\gamma+1)} g(\alpha, \beta, \alpha+\beta-\gamma+1; 1-z), \\ |\arg(1-z)| < \pi, \quad \alpha \text{ and } \beta \neq q, q-1, q-2, \ldots \end{array}\right\} \quad (1)$$

$$\left.\begin{array}{c} g(\alpha, \beta, \gamma; z) = \\ (-1)^{\gamma} \dfrac{\Gamma(\gamma)\,\Gamma(\alpha-\gamma+1)\,\Gamma(\beta-\gamma+1)}{\Gamma(\alpha+\beta-\gamma+1)} F(\alpha, \beta, \alpha+\beta-\gamma+1; 1-z), \\ |\arg z| < \pi, \quad \alpha \text{ and } \beta \neq p, p-1, p-2, \ldots \end{array}\right\} \quad (2)$$

$$\left.\begin{array}{c} g_1(\alpha, \beta, \gamma; z) = -e^{\mp \pi i \alpha} \dfrac{(1-\alpha)_q}{(1-\alpha)_p} \dfrac{p!}{q!} g_1(\alpha, \beta, \alpha+\beta-\gamma+1; 1-z), \\ \alpha \neq 1, 2, 3, \ldots \end{array}\right\} \quad (3)$$

$$\left.\begin{array}{c} g_1(\beta,\alpha,\gamma;z) = -e^{\mp\pi i\beta}\dfrac{(1-\beta)_q}{(1-\beta)_p}\dfrac{p\,!}{q\,!}\,g_1(\beta,\alpha,\alpha+\beta-\gamma+1;1-z), \\ \beta \neq 1,2,3,\ldots \end{array}\right\} \quad (4)$$

$$\left.\begin{array}{c} F(\alpha,\beta,\gamma;z) = (-1)^n\dfrac{(q+1)_n}{(p+1)_n}F(\alpha,\beta,\alpha+\beta-\gamma+1;1-z), \\ \alpha \text{ or } \beta = -n, \quad n = 0,1,2,\ldots \end{array}\right\} \quad (5)$$

$$\left.\begin{array}{c} F(\alpha,\beta,\gamma;z) = \\ \dfrac{\Gamma(\gamma)\,\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)}(1-z)^{\gamma-\alpha-\beta}f(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1;1-z), \\ \alpha \text{ or } \beta = 1,2,\ldots q. \end{array}\right\} \quad (6)$$

$$\left.\begin{array}{c} z^{1-\gamma}f(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;z) = \\ \dfrac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\gamma-1)\,\Gamma(\alpha+\beta-\gamma+1)}F(\alpha,\beta,\alpha+\beta-\gamma+1;1-z), \\ \alpha \text{ or } \beta = 1,2,\ldots p. \end{array}\right\} \quad (7)$$

6. $\gamma = 1+p, \quad \gamma-\alpha-\beta = q.$

$$\left.\begin{array}{c} F(\alpha,\beta,\gamma;z) = \\ \dfrac{(-1)^{\gamma-\alpha-\beta+1}\,\Gamma(\gamma)\,(1-z)^{\gamma-\alpha-\beta}}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma-\alpha-\beta+1)}g(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1;1-z), \\ |\arg(1-z)| < \pi, \quad \alpha \text{ and } \beta \neq 0,-1,-2,\ldots \end{array}\right\} \quad (1)$$

$$\left.\begin{array}{c} g_0(\alpha,\beta,\gamma;z) = \\ (-1)^{\gamma}\dfrac{\Gamma(\gamma)\,\Gamma(1-\alpha)\,\Gamma(1-\beta)}{\Gamma(\gamma-\alpha-\beta+1)}(1-z)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1;1-z), \\ |\arg z| < \pi, \quad \alpha \text{ and } \beta \neq 1,2,3,\ldots \end{array}\right\} \quad (2)$$

$$\left.\begin{array}{c} g_1(\alpha,\beta,\gamma;z) = \\ e^{\pm\pi i\,(\beta+q)}\dfrac{\Gamma(\gamma)\,\Gamma(1-\beta)}{\Gamma(\alpha)\,\Gamma(\gamma-\alpha-\beta+1)}(1-z)^{\gamma-\alpha-\beta}g_1(\gamma-\beta,\gamma-\alpha,\gamma-\alpha-\beta+1;1-z), \\ \beta \neq p,p-1,p-2,\ldots \end{array}\right\} \quad (3)$$

$$\left.\begin{array}{c} g_1(\beta,\alpha,\gamma;z) = \\ e^{\pm\pi i\,(\alpha+q)}\dfrac{\Gamma(\gamma)\,\Gamma(1-\alpha)}{\Gamma(\beta)\,\Gamma(\gamma-\alpha-\beta+1)}(1-z)^{\gamma-\alpha-\beta}g_1(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1;1-z), \\ \alpha \neq p,p-1,p-2,\ldots \end{array}\right\} \quad (4)$$

$$\left.\begin{array}{c} F(\alpha,\beta,\gamma;z) = (-1)^n \dfrac{(1+q)_n}{(1+p)_n}(1-z)^{\gamma-\alpha-\beta} F(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1;1-z), \\ \alpha \text{ or } \beta = \gamma+n, \quad n = 0,1,2,\ldots \end{array}\right\} \quad (5)$$

$$\left.\begin{array}{c} F(\alpha,\beta,\gamma;z) = \dfrac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)} f(\alpha,\beta,\alpha+\beta-\gamma+1;1-z), \\ \alpha \text{ or } \beta = 0,-1,\ldots 1-q. \end{array}\right\} \quad (6)$$

$$\left.\begin{array}{c} z^{1-\gamma}(1-z)^{\gamma-\alpha-\beta} f(1-\alpha,1-\beta,2-\gamma;z) = \\ \dfrac{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)\,(1-z)^{\gamma-\alpha-\beta}}{\Gamma(\gamma-1)\,\Gamma(\gamma-\alpha-\beta+1)} F(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1;1-z), \\ \alpha \text{ or } \beta = 1,2,\ldots p. \end{array}\right\} \quad (7)$$

7. $\gamma = 1-p, \quad \gamma-\alpha-\beta = q.$

$$\left.\begin{array}{c} z^{1-\gamma} F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;z) = \\ \dfrac{(-1)^{q+1}\,\Gamma(2-\gamma)\,(1-z)^{\gamma-\alpha-\beta}}{\Gamma(\alpha-\gamma+1)\,\Gamma(\beta-\gamma+1)\,\Gamma(\gamma-\alpha-\beta+1)} g_0(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1;1-z), \\ |\arg(1-z)| < \pi, \quad \alpha \text{ and } \beta \neq -p,-p-1,-p-2,\ldots \end{array}\right\} \quad (1)$$

$$\left.\begin{array}{c} z^{1-\gamma} g_0(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;z) = \\ (-1)^{\gamma} \dfrac{\Gamma(2-\gamma)\,\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}{\Gamma(\gamma-\alpha-\beta+1)} (1-z)^{\gamma-\alpha-\beta} F(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1;1-z), \\ |\arg z| < \pi, \quad \alpha \text{ and } \beta \neq -q,-q-1,-q-2,\ldots \end{array}\right\} \quad (2)$$

$$\left.\begin{array}{c} z^{1-\gamma} g_1(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;z) = \\ e^{\mp\pi i(\alpha+1)} \dfrac{p!\,(\alpha)_q}{q!\,(\alpha)_p} (1-z)^{\gamma-\alpha-\beta} g_1(\gamma-\beta,\gamma-\alpha,\gamma-\alpha-\beta+1;1-z), \\ \beta \neq 0,-1,-2,\ldots \end{array}\right\} \quad (3)$$

$$\left.\begin{array}{c} z^{1-\gamma} g_1(\beta-\gamma+1,\alpha-\gamma+1,2-\gamma;z) = \\ e^{\mp\pi i(\beta+1)} \dfrac{p!\,(\beta)_q}{q!\,(\beta)_p} (1-z)^{\gamma-\alpha-\beta} g_1(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1;1-z), \\ \alpha \neq 0,-1,-2,\ldots \end{array}\right\} \quad (4)$$

$$\left.\begin{array}{c} z^{1-\gamma} F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;z) = \\ (-1)^n \dfrac{(q+1)_n}{(p+1)_n} (1-z)^{\gamma-\alpha-\beta} F(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1;1-z), \\ \alpha \text{ or } \beta = n+1, \quad n = 0,1,2,\ldots \end{array}\right\} \quad (5)$$

$$\left.\begin{array}{c}(1-z)^{\gamma-\alpha-\beta} f(\gamma-\alpha, \gamma-\beta, \gamma; z) = \\ \dfrac{\Gamma(1-\alpha)\,\Gamma(1-\beta)\,(1-z)^{\gamma-\alpha-\beta}}{\Gamma(1-\gamma)\,\Gamma(\gamma-\alpha-\beta+1)} F(\gamma-\alpha, \gamma-\beta, \gamma-\alpha-\beta+1; 1-z), \\ \alpha \text{ or } \beta = 0, -1, \ldots 1-p.\end{array}\right\} \quad (6)$$

$$\left.\begin{array}{c}z^{1-\gamma} F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma; z) = \\ \dfrac{\Gamma(2-\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(1-\alpha)\,\Gamma(1-\beta)} z^{1-\gamma} f(\alpha-\gamma+1, \beta-\gamma+1, \alpha+\beta-\gamma+1; 1-z), \\ \alpha \text{ or } \beta = 0, -1, \ldots 1-q.\end{array}\right\} \quad (7)$$

8. $\gamma = 1-p, \quad \alpha+\beta-\gamma = q.$

$$\left.\begin{array}{c}z^{1-\gamma} F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma; z) = \\ \dfrac{(-1)^{\alpha+\beta-\gamma+1}\,\Gamma(2-\gamma)}{\Gamma(1-\alpha)\,\Gamma(1-\beta)\,\Gamma(\alpha+\beta-\gamma+1)} g_0(\alpha, \beta, \alpha+\beta-\gamma+1; 1-z), \\ |\arg(1-z)| < \pi, \quad \alpha \text{ and } \beta \neq 1, 2, 3, \ldots\end{array}\right\} \quad (1)$$

$$\left.\begin{array}{c}z^{1-\gamma} g(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma; z) = \\ (-1)^{\gamma} \dfrac{\Gamma(2-\gamma)\,\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta-\gamma+1)} F(\alpha, \beta, \alpha+\beta-\gamma+1; 1-z), \\ |\arg z| < \pi, \quad \alpha \text{ and } \beta \neq 0, -1, -2, \ldots\end{array}\right\} \quad (2)$$

$$\left.\begin{array}{c}z^{1-\gamma} g_1(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma; z) = \\ -\dfrac{e^{\mp \pi i(\alpha+q)}\,\Gamma(\alpha)\,p!}{\Gamma(\alpha+p-q)\,q!} g_1(\alpha, \beta, \alpha+\beta-\gamma+1; 1-z), \\ \alpha \neq 1-p, 2-p, 3-p, \ldots\end{array}\right\} \quad (3)$$

$$\left.\begin{array}{c}z^{1-\gamma} g_1(\beta-\gamma+1, \alpha-\gamma+1, 2-\gamma; z) = \\ -\dfrac{e^{\mp \pi i(\beta+q)}\,\Gamma(\beta)\,p!}{\Gamma(\beta+p-q)\,q!} g_1(\beta, \alpha, \alpha+\beta-\gamma+1; 1-z), \\ \beta \neq 1-p, 2-p, 3-p, \ldots\end{array}\right\} \quad (4)$$

$$\left.\begin{array}{c}z^{1-\gamma} F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma; z) = \\ (-1)^n \dfrac{(q+1)_n}{(p+1)_n} F(\alpha, \beta, \alpha+\beta-\gamma+1; 1-z), \\ \alpha \text{ or } \beta = -p-n, \quad n = 0, 1, 2, \ldots\end{array}\right\} \quad (5)$$

$$\left.\begin{array}{c} f(\alpha,\beta,\gamma;z)=\dfrac{\Gamma(\alpha-\gamma+1)\,\Gamma(\beta-\gamma+1)}{\Gamma(1-\gamma)\,\Gamma(\alpha+\beta-\gamma+1)}\,F(\alpha,\beta,\alpha+\beta-\gamma+1;1-z),\\ \alpha \text{ or } \beta=0,-1,\ldots 1-p. \end{array}\right\}\quad(6)$$

$$\left.\begin{array}{c} z^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;z)=\\ \dfrac{\Gamma(2-\gamma)\,\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha-\gamma+1)\,\Gamma(\beta-\gamma+1)}\,z^{1-\gamma}(1-z)^{\gamma-\alpha-\beta}f(1-\alpha,1-\beta,\gamma-\alpha-\beta+1;1-z),\\ \alpha \text{ or } \beta=1,2,\ldots q. \end{array}\right\}\quad(7)$$

9. $\beta=\alpha+m,\quad \gamma$ not an integer.

$$\left.\begin{array}{c} F(\alpha,\beta,\gamma;z)=\dfrac{(-1)^{m-1}\,\Gamma(\gamma)\,(-z)^{-\beta}}{\Gamma(\alpha)\,\Gamma(\gamma-\beta)\,\Gamma(\beta-\alpha+1)}\,g_1\!\left(\beta-\gamma+1,\beta,\beta-\alpha+1;\dfrac{1}{z}\right),\\ |\arg(-z)|<\pi,\quad \alpha \text{ and } \gamma-\beta\neq 0,-1,-2,\ldots \end{array}\right\}\quad(1)$$

$$\left.\begin{array}{c} F(\alpha,\beta,\gamma;z)=\dfrac{\Gamma(1-\alpha)\,\Gamma(\beta-\gamma+1)}{\Gamma(1-\gamma)\,\Gamma(\beta-\alpha+1)}\,(-z)^{-\beta}F\!\left(\beta,\beta-\gamma+1,\beta-\alpha+1;\dfrac{1}{z}\right),\\ \gamma-\alpha \text{ or } \beta=0,-1,-2,\ldots \end{array}\right\}\quad(2)$$

$$\left.\begin{array}{c} F(\alpha,\beta,\gamma;z)=\dfrac{\Gamma(\gamma)\,\Gamma(\beta-\alpha)}{\Gamma(\beta)\,\Gamma(\gamma-\alpha)}\,(-z)^{-\alpha}f\!\left(\alpha,\alpha-\gamma+1,\alpha-\beta+1;\dfrac{1}{z}\right),\\ \beta=1,2,\ldots m. \end{array}\right\}\quad(3)$$

$$\left.\begin{array}{c} F(\alpha,\beta,\gamma;z)=\dfrac{\Gamma(\gamma)\,\Gamma(\beta-\alpha)}{\Gamma(\beta)\,\Gamma(\gamma-\alpha)}\,(-z)^{-\alpha}\left(1-\dfrac{1}{z}\right)^{\gamma-\alpha-\beta}f\!\left(1-\beta,\gamma-\beta,\alpha-\beta+1;\dfrac{1}{z}\right),\\ \beta-\gamma=0,1,2,\ldots m-1. \end{array}\right\}\quad(4)$$

$$\left.\begin{array}{c} (-z)^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;z)=\\ \dfrac{(-1)^{m-1}\,\Gamma(2-\gamma)\,(-z)^{-\beta}}{\Gamma(1-\beta)\,\Gamma(\alpha-\gamma+1)\,\Gamma(\beta-\alpha+1)}\,g_1\!\left(\beta,\beta-\gamma+1,\beta-\alpha+1;\dfrac{1}{z}\right),\\ |\arg(-z)|<\pi,\quad \beta \text{ and } \gamma-\alpha\neq 1,2,3,\ldots \end{array}\right\}\quad(5)$$

$$\left.\begin{array}{c} (-z)^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;z)=\\ \dfrac{\Gamma(\beta)\,\Gamma(\gamma-\alpha)}{\Gamma(\gamma-1)\,\Gamma(\beta-\alpha+1)}\,(-z)^{-\beta}F\!\left(\beta,\beta-\gamma+1,\beta-\alpha+1;\dfrac{1}{z}\right),\\ \alpha \text{ or } \gamma-\beta=1,2,3,\ldots \end{array}\right\}\quad(6)$$

$$\left.\begin{array}{c}(-z)^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;z)=\\ \dfrac{\Gamma(2-\gamma)\,\Gamma(\beta-\alpha)}{\Gamma(\beta-\gamma+1)\,\Gamma(1-\alpha)}(-z)^{\alpha}f\left(\alpha,\,\alpha-\gamma+1,\,\alpha-\beta+1;\dfrac{1}{z}\right),\\ \beta-\gamma=0,1,\ldots m-1.\end{array}\right\}\quad(7)$$

$$\left.\begin{array}{c}(-z)^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;z)=\\ \dfrac{\Gamma(2-\gamma)\,\Gamma(\beta-\alpha)}{\Gamma(\beta-\gamma+1)\,\Gamma(1-\alpha)}(-z)^{-\alpha}\left(1-\dfrac{1}{z}\right)^{\gamma-\alpha-\beta}f\left(1-\beta,\gamma-\beta,\,\alpha-\beta+1;\dfrac{1}{z}\right),\\ \beta=1,2,\ldots m.\end{array}\right\}\quad(8)$$

10. $\gamma=1+p,\quad \alpha-\beta$ non-integer.

$$\left.\begin{array}{c}F(\alpha,\beta,\gamma;z)=\dfrac{\Gamma(\gamma)\,\Gamma(\beta-\alpha)}{\Gamma(\beta)\,\Gamma(\gamma-\alpha)}(-z)^{-\alpha}F\left(\alpha,\,\alpha-\gamma+1,\,\alpha-\beta+1;\dfrac{1}{z}\right)+\\ \dfrac{\Gamma(\gamma)\,\Gamma(\alpha-\beta)}{\Gamma(\alpha)\,\Gamma(\gamma-\beta)}(-z)^{-\beta}F\left(\beta,\beta-\gamma+1,\,\beta-\alpha+1;\dfrac{1}{z}\right),\\ |\arg(-z)|<\pi.\end{array}\right\}\quad(1)$$

$$\left.\begin{array}{c}g_1(\alpha,\beta,\gamma;z)=\\ (-1)^{\gamma}\dfrac{\Gamma(\gamma)\,\Gamma(1-\alpha)\,\Gamma(\beta-\gamma+1)}{\Gamma(\beta-\alpha+1)}(-z)^{-\beta}F\left(\beta,\beta-\gamma+1,\,\beta-\alpha+1;\dfrac{1}{z}\right),\\ |\arg(-z)|<\pi,\quad \alpha\text{ and }\gamma-\beta\neq1,2,3,\ldots\end{array}\right\}\quad(2)$$

$$\left.\begin{array}{c}g_1(\beta,\alpha,\gamma;z)=\\ (-1)^{\gamma}\dfrac{\Gamma(\gamma)\,\Gamma(1-\beta)\,\Gamma(\alpha-\gamma+1)}{\Gamma(\alpha-\beta+1)}(-z)^{-\alpha}F\left(\alpha,\,\alpha-\gamma+1,\,\alpha-\beta+1;\dfrac{1}{z}\right),\\ |\arg(-z)|<\pi,\quad \beta\text{ and }\gamma-\alpha\neq1,2,3,\ldots\end{array}\right\}\quad(3)$$

$$\left.\begin{array}{c}F(\alpha,\beta,\gamma;z)=\dfrac{\Gamma(\gamma)\,\Gamma(\beta-\alpha)}{\Gamma(\beta)\,\Gamma(\gamma-\alpha)}(-z)^{-\alpha}F\left(\alpha,\,\alpha-\gamma+1,\,\alpha-\beta+1;\dfrac{1}{z}\right),\\ \alpha\text{ or }\gamma-\beta=0,-1,-2,\ldots\end{array}\right\}\quad(4)$$

$$\left.\begin{array}{c}F(\alpha,\beta,\gamma;z)=\dfrac{\Gamma(\gamma)\,\Gamma(\alpha-\beta)}{\Gamma(\alpha)\,\Gamma(\gamma-\beta)}(-z)^{-\beta}F\left(\beta,\beta-\gamma+1,\,\beta-\alpha+1;\dfrac{1}{z}\right),\\ \beta\text{ or }\gamma-\alpha=0,-1,-2,\ldots\end{array}\right\}\quad(5)$$

$$\left.\begin{array}{c}(-z)^{1-\gamma} f(\alpha-\gamma+1\,,\,\beta-\gamma+1\,,\,2-\gamma;\,z) = \\ \dfrac{\Gamma(\alpha)\,\Gamma(\gamma-\beta)}{\Gamma(\gamma-1)\,\Gamma(\alpha-\beta+1)}(-z)^{-\alpha} F\left(\alpha,\,\alpha-\gamma+1\,,\,\alpha-\beta+1\,;\dfrac{1}{z}\right), \\ \alpha = 1\,,\,2,\,\ldots\, p.\end{array}\right\} \quad (6)$$

$$\left.\begin{array}{c}(-z)^{1-\gamma} f(\alpha-\gamma+1\,,\,\beta-\gamma+1\,,\,2-\gamma;\,z) = \\ \dfrac{\Gamma(\beta)\,\Gamma(\gamma-\alpha)}{\Gamma(\gamma-1)\,\Gamma(\beta-\alpha+1)}(-z)^{-\beta} F\left(\beta,\,\beta-\gamma+1\,,\,\beta-\alpha+1\,;\dfrac{1}{z}\right), \\ \beta = 1\,,\,2,\,\ldots\, p.\end{array}\right\} \quad (7)$$

$$\left.\begin{array}{c}(-z)^{1-\gamma}(1-z)^{\gamma-\alpha-\beta} f(1-\alpha,\,1-\beta,\,2-\gamma;\,z) = \\ \dfrac{\Gamma(\beta)\,\Gamma(\gamma-\alpha)}{\Gamma(\gamma-1)\,\Gamma(\beta-\alpha+1)}(-z)^{-\beta} F\left(\beta,\,\beta-\gamma+1\,,\,\beta-\alpha+1\,;\dfrac{1}{z}\right), \\ \alpha = 1\,,\,2,\,\ldots\, p.\end{array}\right\} \quad (8)$$

$$\left.\begin{array}{c}(-z)^{1-\gamma}(1-z)^{\gamma-\alpha-\beta} f(1-\alpha,\,1-\beta,\,2-\gamma;\,z) = \\ \dfrac{\Gamma(\alpha)\,\Gamma(\gamma-\beta)}{\Gamma(\gamma-1)\,\Gamma(\alpha-\beta+1)}(-z)^{-\alpha} F\left(\alpha,\,\alpha-\gamma+1\,,\,\alpha-\beta+1\,;\dfrac{1}{z}\right), \\ \beta = 1\,,\,2,\,\ldots\, p.\end{array}\right\} \quad (9)$$

11. $\gamma = 1-p, \quad \alpha-\beta$ non-integer.

$$\left.\begin{array}{c}(-z)^{1-\gamma} F(\alpha-\gamma+1\,,\,\beta-\gamma+1\,,\,2-\gamma;\,z) = \\ \dfrac{\Gamma(2-\gamma)\,\Gamma(\beta-\alpha)}{\Gamma(1-\alpha)\,\Gamma(\beta-\gamma+1)}(-z)^{-\alpha} F\left(\alpha,\,\alpha-\gamma+1\,,\,\alpha-\beta+1\,;\dfrac{1}{z}\right) + \\ \dfrac{\Gamma(2-\gamma)\,\Gamma(\alpha-\beta)}{\Gamma(1-\beta)\,\Gamma(\alpha-\gamma+1)}(-z)^{-\beta} F\left(\beta,\,\beta-\gamma+1\,,\,\beta-\alpha+1\,;\dfrac{1}{z}\right), \\ |\arg(-z)| < \pi.\end{array}\right\} \quad (1)$$

$$\left.\begin{array}{c}(-z)^{1-\gamma} g_1(\alpha-\gamma+1\,,\,\beta-\gamma+1\,,\,2-\gamma;\,z) = \\ (-1)^{\gamma}\dfrac{\Gamma(2-\gamma)\,\Gamma(\gamma-\alpha)\,\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}(-z)^{-\beta} F\left(\beta,\,\beta-\gamma+1\,,\,\beta-\alpha+1\,;\dfrac{1}{z}\right), \\ |\arg(-z)| < \pi, \quad \beta \text{ and } \gamma-\alpha \neq 0,\,-1,\,-2,\,\ldots\end{array}\right\} \quad (2)$$

$$\left.\begin{array}{c}(-z)^{1-\gamma} g_1(\beta-\gamma+1\,,\,\alpha-\gamma+1\,,\,2-\gamma;\,z) = \\ (-1)^{\gamma}\dfrac{\Gamma(2-\gamma)\,\Gamma(\gamma-\beta)\,\Gamma(\alpha)}{\Gamma(\alpha-\beta+1)}(-z)^{-\alpha} F\left(\alpha,\,\alpha-\gamma+1\,,\,\alpha-\beta+1\,;\dfrac{1}{z}\right), \\ \alpha \text{ and } \gamma-\beta \neq 0,\,-1,\,-2,\,\ldots\end{array}\right\} \quad (3)$$

$$\left.\begin{aligned}&(-z)^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;z)=\\&\frac{\Gamma(2-\gamma)\,\Gamma(\beta-\alpha)}{\Gamma(1-\alpha)\,\Gamma(\beta-\gamma+1)}(-z)^{-\alpha}F\left(\alpha,\,\alpha-\gamma+1,\,\alpha-\beta+1;\frac{1}{z}\right),\\&\beta\text{ or }\gamma-\alpha=1,2,3,\ldots\end{aligned}\right\}\quad(4)$$

$$\left.\begin{aligned}&(-z)^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;z)=\\&\frac{\Gamma(2-\gamma)\,\Gamma(\alpha-\beta)}{\Gamma(1-\beta)\,\Gamma(\alpha-\gamma+1)}(-z)^{-\beta}F\left(\beta,\,\beta-\gamma+1,\,\beta-\alpha+1;\frac{1}{z}\right),\\&\alpha\text{ or }\gamma-\beta=1,2,3,\ldots\end{aligned}\right\}\quad(5)$$

$$\left.\begin{aligned}&f(\alpha,\beta,\gamma;z)=\frac{\Gamma(\alpha-\gamma+1)\,\Gamma(1-\beta)}{\Gamma(1-\gamma)\,\Gamma(\alpha-\beta+1)}(-z)^{-\alpha}F\left(\alpha,\,\alpha-\gamma+1,\,\alpha-\beta+1;\frac{1}{z}\right),\\&\alpha=0,-1,\ldots 1-p.\end{aligned}\right\}\quad(6)$$

$$\left.\begin{aligned}&f(\alpha,\beta,\gamma;z)=\frac{\Gamma(\beta-\gamma+1)\,\Gamma(1-\alpha)}{\Gamma(1-\gamma)\,\Gamma(\beta-\alpha+1)}(-z)^{-\beta}F\left(\beta,\,\beta-\gamma+1,\,\beta-\alpha+1;\frac{1}{z}\right),\\&\beta=0,-1,\ldots 1-p.\end{aligned}\right\}\quad(7)$$

$$\left.\begin{aligned}&(1-z)^{\gamma-\alpha-\beta}f(\gamma-\alpha,\gamma-\beta,\gamma;z)=\\&\frac{\Gamma(\beta-\gamma+1)\,\Gamma(1-\alpha)}{\Gamma(1-\gamma)\,\Gamma(\beta-\alpha+1)}(-z)^{-\beta}F\left(\beta,\,\beta-\gamma+1,\,\beta-\alpha+1;\frac{1}{z}\right),\\&\alpha=0,-1,\ldots 1-p.\end{aligned}\right\}\quad(8)$$

$$\left.\begin{aligned}&(1-z)^{\gamma-\alpha-\beta}f(\gamma-\alpha,\gamma-\beta,\gamma;z)=\\&\frac{\Gamma(\alpha-\gamma+1)\,\Gamma(1-\beta)}{\Gamma(1-\gamma)\,\Gamma(\alpha-\beta+1)}(-z)^{-\alpha}F\left(\alpha,\,\alpha-\gamma+1,\,\alpha-\beta+1;\frac{1}{z}\right),\\&\beta=0,-1,\ldots 1-p.\end{aligned}\right\}\quad(9)$$

12. $\gamma=1+p,\quad \beta=\alpha+m.$

$$\left.\begin{aligned}&F(\alpha,\beta,\gamma;z)=\frac{(-1)^{m+1}\,\Gamma(\gamma)\,(-z)^{-\beta}}{\Gamma(\alpha)\,\Gamma(\gamma-\beta)\,\Gamma(\beta-\alpha+1)}g_1\left(\beta-\gamma+1,\,\beta,\,\beta-\alpha+1;\frac{1}{z}\right),\\&|\arg(-z)|<\pi,\quad \alpha\text{ and }\gamma-\beta\neq 0,-1,-2,\ldots\end{aligned}\right\}\quad(1)$$

$$\left.\begin{aligned}&g_1(\alpha,\beta,\gamma;z)=\\&(-1)^{\gamma}\frac{\Gamma(\gamma)\,\Gamma(1-\alpha)\,\Gamma(\beta-\gamma+1)}{\Gamma(\beta-\alpha+1)}(-z)^{-\beta}F\left(\beta,\,\beta-\gamma+1,\,\beta-\alpha+1;\frac{1}{z}\right),\\&|\arg(-z)|<\pi,\quad \alpha\text{ and }\gamma-\beta\neq 1,2,3,\ldots\end{aligned}\right\}\quad(2)$$

$$\left.\begin{array}{c} g(\alpha,\beta,\gamma;z) = (-1)^{m-p}\dfrac{\Gamma(\gamma)\,\Gamma(\beta-\gamma+1)}{\Gamma(\alpha)\,\Gamma(\beta-\alpha+1)}\,z^{-\beta}\,g\left(\beta,\,\beta-\gamma+1,\,\beta-\alpha+1;\dfrac{1}{z}\right), \\ |\arg z| < \pi, \quad \gamma-\alpha \neq 1,2,3,\ldots \end{array}\right\} \tag{3}$$

$$\left.\begin{array}{c} g_0(\alpha,\beta,\gamma;z) = \\ -e^{\pm\pi i\beta}\dfrac{\Gamma(\gamma)\,\Gamma(1-\alpha)}{\Gamma(\gamma-\beta)\,\Gamma(\beta-\alpha+1)}(-z)^{-\beta}\,g_0\left(\beta,\,\beta-\gamma+1,\,\beta-\alpha+1;\dfrac{1}{z}\right), \\ \beta \neq 1,2,3,\ldots \end{array}\right\} \tag{4}$$

$$\left.\begin{array}{c} F(\alpha,\beta,\gamma;z) = (-1)^{\varepsilon}\dfrac{p!\,(1-\beta)_m}{m!\,(1-\beta)_p}\,z^{-\beta}\,F\left(\beta,\,\beta-\gamma+1,\,\beta-\alpha+1;\dfrac{1}{z}\right), \\ \text{provided that} \quad \beta = 0,-1,-2,\ldots \quad \text{and} \quad \varepsilon = 0, \\ \text{or} \quad \beta = p+m+1,\,p+m+2,\ldots \quad \text{and} \quad \varepsilon = 1. \end{array}\right\} \tag{5}$$

$$\left.\begin{array}{c} (-z)^{1-\gamma}f(\alpha-\gamma+1,\,\beta-\gamma+1,\,2-\gamma;z) = \\ \dfrac{\Gamma(\beta)\,\Gamma(\gamma-\alpha)}{\Gamma(\gamma-1)\,\Gamma(\beta-\alpha+1)}(-z)^{-\beta}\,F\left(\beta,\,\beta-\gamma+1,\,\beta-\alpha+1;\dfrac{1}{z}\right), \\ \beta = 1,2,\ldots p. \end{array}\right\} \tag{6}$$

$$\left.\begin{array}{c} F(\alpha,\beta,\gamma;z) = \\ \dfrac{\Gamma(\gamma)\,\Gamma(\beta-\alpha)}{\Gamma(\gamma-\alpha)\,\Gamma(\beta)}(-z)^{-\alpha}\,f\left(\alpha,\,\alpha-\gamma+1,\,\alpha-\beta+1;\dfrac{1}{z}\right), \\ \beta = 1,2,\ldots m. \end{array}\right\} \tag{7}$$

$$\left.\begin{array}{c} F(\alpha,\beta,\gamma;z) = \\ \dfrac{\Gamma(\gamma)\,\Gamma(\beta-\alpha)}{\Gamma(\gamma-\alpha)\,\Gamma(\beta)}(-z)^{-\alpha}\left(1-\dfrac{1}{z}\right)^{\gamma-\alpha-\beta}f\left(1-\beta,\,\gamma-\beta,\,\alpha-\beta+1;\dfrac{1}{z}\right), \\ \beta = p+1,\,p+2,\ldots p+m. \end{array}\right\} \tag{8}$$

$$\left.\begin{array}{c} z^{1-\gamma}f(\alpha-\gamma+1,\,\beta-\gamma+1,\,2-\gamma;z) = \\ \dfrac{\Gamma(\alpha)\,\Gamma(\beta-\alpha)}{\Gamma(\gamma-1)\,\Gamma(\beta-\gamma+1)}\,z^{-\alpha}f\left(\alpha,\,\alpha-\gamma+1,\,\alpha-\beta+1;\dfrac{1}{z}\right), \\ \beta = p+1,\,p+2,\ldots p+m \quad \text{if} \quad p \geqq m, \\ \beta = m+1,\,m+2,\ldots m+p \quad \text{if} \quad p < m. \end{array}\right\} \tag{9}$$

13. $\gamma = 1 - p, \quad \beta = \alpha + m.$

$$\left.\begin{aligned}&(-z)^{1-\gamma} F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma; z) = \\ &\frac{(-1)^{m+1}\Gamma(2-\gamma)(-z)^{-\beta}}{\Gamma(\alpha-\gamma+1)\Gamma(1-\beta)\Gamma(\beta-\alpha+1)} g_1\left(\beta, \beta-\gamma+1, \beta-\alpha+1; \frac{1}{z}\right), \\ &|\arg(-z)| < \pi, \quad \beta \text{ and } \gamma-\alpha \neq 1, 2, 3, \ldots\end{aligned}\right\} \quad (1)$$

$$\left.\begin{aligned}&(-z)^{1-\gamma} g_1(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma; z) = \\ &(-1)^{\gamma}\frac{\Gamma(2-\gamma)\Gamma(\gamma-\alpha)\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}(-z)^{-\beta} F\left(\beta, \beta-\gamma+1, \beta-\alpha+1; \frac{1}{z}\right), \\ &|\arg(-z)| < \pi, \quad \beta \text{ and } \gamma-\alpha \neq 0, -1, -2, \ldots\end{aligned}\right\} \quad (2)$$

$$\left.\begin{aligned}&z^{1-\gamma} g(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma; z) = \\ &\frac{(-1)^{m-p}\Gamma(2-\gamma)\Gamma(\beta)}{\Gamma(\alpha-\gamma+1)\Gamma(\beta-\alpha+1)} z^{-\beta} g\left(\beta, \beta-\gamma+1, \beta-\alpha+1; \frac{1}{z}\right), \\ &|\arg z| < \pi, \quad \alpha \neq 0, -1, -2, \ldots\end{aligned}\right\} \quad (3)$$

$$\left.\begin{aligned}&z^{1-\gamma} g_0(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma; z) = \\ &-e^{\pm\pi i\beta}\frac{\Gamma(2-\gamma)\Gamma(\gamma-\alpha)}{\Gamma(1-\beta)\Gamma(\beta-\alpha+1)}(-z)^{-\beta} g_0\left(\beta, \beta-\gamma+1, \beta-\alpha+1; \frac{1}{z}\right), \\ &\gamma-\beta \neq 0, -1, -2, \ldots\end{aligned}\right\} \quad (4)$$

$$\left.\begin{aligned}&(-z)^{1-\gamma} F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma; z) = \\ &\frac{(-1)^{m+\varepsilon}p!}{m!(\beta)_{p-m}} z^{-\beta} F\left(\beta, \beta-\gamma+1, \beta-\alpha+1; \frac{1}{z}\right), \\ &\text{provided that } \beta = -p, -p-1, -p-2, \ldots \text{ and } \varepsilon = 0, \\ &\text{or } \beta = m+1, m+2, m+3, \ldots \text{ and } \varepsilon = 1.\end{aligned}\right\} \quad (5)$$

$$\left.\begin{aligned}&f(\alpha, \beta, \gamma; z) = \\ &\frac{\Gamma(\beta-\gamma+1)\Gamma(1-\alpha)}{\Gamma(1-\gamma)\Gamma(\beta-\alpha+1)}(-z)^{-\beta} F\left(\beta, \beta-\gamma+1, \beta-\alpha+1; \frac{1}{z}\right), \\ &\beta = 0, -1, -2, \ldots 1-p.\end{aligned}\right\} \quad (6)$$

$$\left.\begin{aligned}&(-z)^{1-\gamma} F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma; z) = \\ &\frac{\Gamma(2-\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta-\gamma+1)\Gamma(1-\alpha)}(-z)^{-\alpha} f\left(\alpha, \alpha-\gamma+1, \alpha-\beta+1; \frac{1}{z}\right), \\ &\beta-\gamma = 0, 1, 2, \ldots m-1.\end{aligned}\right\} \quad (7)$$

$$\left.\begin{array}{c}(-z)^{1-\gamma} F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma; z) = \\ \dfrac{\Gamma(2-\gamma)\,\Gamma(\beta-\alpha)}{\Gamma(\beta-\gamma+1)\,\Gamma(1-\alpha)}(-z)^{-\alpha}\left(1-\dfrac{1}{z}\right)^{\gamma-\alpha-\beta} f\left(1-\beta, \gamma-\beta, \alpha-\beta+1; \dfrac{1}{z}\right), \\ \beta = 1, 2, 3, \ldots m.\end{array}\right\} \quad (8)$$

$$\left.\begin{array}{l}f(\alpha, \beta, \gamma; z) = \\ \dfrac{\Gamma(\alpha-\gamma+1)\,\Gamma(\beta-\alpha)}{\Gamma(1-\gamma)\,\Gamma(\beta)} z^{-\alpha} f\left(\alpha, \alpha-\gamma+1, \alpha-\beta+1; \dfrac{1}{z}\right) \\ \beta = 1, 2, 3, \ldots m \qquad \text{if} \quad p \geqq m, \\ \beta = m-p+1, m-p+2, \ldots m \quad \text{if} \quad p < m.\end{array}\right\} \quad (9)$$

Chapter II

Quadratic Transformations

§ 11. Quadratic transformations of the hypergeometric function $F(\alpha, \beta, \gamma; z)$ have been investigated by GAUSS and in a more complete manner by KUMMER and GOURSAT. We shall now give some examples of such transformations of the logarithmic and other exceptional solutions. We take the cases considered by GAUSS. If we put

$$x = \frac{4z}{(1+z)^2}, \qquad Y = (1+z)^{2\alpha} y,$$

the equation

$$x(1-x)\frac{d^2Y}{dx^2} + [\gamma - (\alpha+\beta+1)x]\frac{dY}{dx} - \alpha\beta Y = 0 \quad (65)$$

is transformed into

$$\left.\begin{array}{l}z(1-z^2)\dfrac{d^2y}{dz^2} + [\gamma - (4\beta - 2\gamma)z + (\gamma - 4\alpha - 2)z^2]\dfrac{dy}{dz} \\ \qquad - 2\alpha[2\beta - \gamma + (2\alpha - \gamma + 1)z]\,y = 0.\end{array}\right\} \quad (66)$$

If $\beta = \alpha + \frac{1}{2}$ this equation reduces to

$$z(1-z)\frac{d^2y}{dz^2} + [\gamma - (4\alpha - \gamma + 2)z]\frac{dy}{dz} - 2\alpha(2\alpha - \gamma + 1)\,y = 0. \quad (67)$$

It has the solution $F(2\alpha, 2\alpha-\gamma+1, \gamma; z)$ and consequently

$$F\left(\alpha, \alpha+\frac{1}{2}, \gamma; \frac{4z}{(1+z)^2}\right) = (1+z)^{2\alpha} F(2\alpha, 2\alpha-\gamma+1, \gamma; z), \quad (68)$$

as shown by GAUSS. If γ is a positive integer, (65) has the solution $G\left(\alpha, \alpha+\frac{1}{2}, \gamma; x\right)$ and (67) has the second solution $G(2\alpha, 2\alpha-\gamma+1, \gamma; z)$, provided that neither α nor β is one of the numbers $1, 2, \ldots \gamma-1$. It follows that

$$\left.\begin{aligned} G\left(\alpha, \alpha+\frac{1}{2}, \gamma; \frac{4z}{(1+z)^2}\right) = (1+z)^{2\alpha}\,[C_1 G(2\alpha, 2\alpha-\gamma+1, \gamma; z) + \\ C\ F(2\alpha, 2\alpha-\gamma+1, \gamma; z)], \end{aligned}\right\} \tag{69}$$

where C and C_1 are constants. We divide both sides of (69) by $z^{1-\gamma}$ or by $\log z$, if $\gamma = 1$. Now let $z \to 0$, and using (60) or (61), we find that $C_1 = 1$. Next we let $z \to 1$, and using (54), we obtain, if $\Re(\gamma-2\alpha) > \frac{1}{2}$,

$$C = \Psi(\gamma-2\alpha) + \Psi(1-2\alpha) - \Psi(1-\alpha) - \Psi\left(\frac{1}{2}-\alpha\right). \tag{70}$$

If $\Re(\gamma-2\alpha) < \frac{1}{2}$, we use (55) and (56) and we get in the same manner

$$C = \Psi(2\alpha-\gamma+1) + \Psi(2\alpha) - \Psi(\alpha) - \Psi\left(\frac{1}{2}+\alpha\right). \tag{71}$$

Now we have the relation

$$2\Psi(2\alpha) - \Psi(\alpha) - \Psi\left(\alpha+\frac{1}{2}\right) = \log 4.$$

(71) therefore reduces to

$$C = \Psi(2\alpha-\gamma+1) - \Psi(2\alpha) + \log 4 = \sum_{s=1}^{\gamma-1} \frac{1}{s-2\alpha} + \log 4. \tag{72}$$

In a similar manner we get from (70)

$$C = \Psi(\gamma-2\alpha) - \Psi(1-2\alpha) + \log 4 = \sum_{s=1}^{\gamma-1} \frac{1}{s-2\alpha} + \log 4.$$

The relation (69) can therefore be written

$$\left.\begin{aligned} G\left(\alpha, \alpha+\frac{1}{2}, \gamma; \frac{4z}{(1+z)^2}\right) = (1+z)^{2\alpha}\,G(2\alpha, 2\alpha-\gamma+1, \gamma; z) \\ + C\,F\left(\alpha, \alpha+\frac{1}{2}, \gamma; \frac{4z}{(1+z)^2}\right), \end{aligned}\right\} \tag{73}$$

where C is defined by (72) and $\alpha \neq \frac{1}{2}, 1, \frac{3}{2}, \ldots \gamma-1$.

Using (20) or (21), we get from (73)

$$\left.\begin{aligned} g\left(\alpha, \alpha+\frac{1}{2}, \gamma; \frac{4z}{(1+z)^2}\right) = (1+z)^{2\alpha}\,g(2\alpha, 2\alpha-\gamma+1, \gamma; z) \\ = \left(1+\frac{1}{z}\right)^{2\alpha} g\left(2\alpha, 2\alpha-\gamma+1, \gamma; \frac{1}{z}\right), \end{aligned}\right\} \tag{74}$$

provided that 2α is not an integer $<2\gamma-1$, and

$$g_0\left(\alpha,\alpha+\frac{1}{2},\gamma;\frac{4z}{(1+z)^2}\right)=(1+z)^{2\alpha}g_0(2\alpha,2\alpha-\gamma+1,\gamma;z), \tag{75}$$

provided that 2α is not a positive integer. Furthermore using (24) and Table 12, we find that

$$\left.\begin{aligned}(1+z)^{-2\alpha}g_1\left(\alpha,\alpha+\frac{1}{2},\gamma;\frac{4z}{(1+z)^2}\right)=\\ g_1(2\alpha,2\alpha-\gamma+1,\gamma;z)+\frac{\pi}{\sin 2\pi\alpha}F(2\alpha,2\alpha-\gamma+1,\gamma;z)=\\ g_1(2\alpha,2\alpha-\gamma+1,\gamma;z)-(-z)^{-2\alpha}g_1\left(2\alpha,2\alpha-\gamma+1,\gamma;\frac{1}{z}\right)=\\ \left[F(2\alpha,2\alpha-\gamma+1,\gamma;z)-(-z)^{-2\alpha}F\left(2\alpha,2\alpha-\gamma+1,\gamma;\frac{1}{z}\right)\right]\frac{\pi}{\sin 2\pi\alpha},\end{aligned}\right\} \tag{76}$$

provided that 2α is not an integer.

We now consider the rational solutions. The differential equation (65) has the solution $f(\alpha,\alpha+\frac{1}{2},\gamma;x)$, and (67) has the solutions $f(2\alpha,2\alpha-\gamma+1,\gamma;z)$ and $z^{1-\gamma}F(2\alpha-\gamma+1,2\alpha-2\gamma+2,2-\gamma;z)$, if γ is a non-positive integer, and α has one of the values $0,-\frac{1}{2},-1,-\frac{3}{2},\ldots\gamma-\frac{1}{2}$. Hence

$$\left.\begin{aligned}(1+z)^{-2\alpha}f\left(\alpha,\alpha+\frac{1}{2},\gamma;\frac{4z}{(1+z)^2}\right)&=C_1f(2\alpha,2\alpha-\gamma+1,\gamma;z)\\ &+Cz^{1-\gamma}F(2\alpha-\gamma+1,2\alpha-2\gamma+2,2-\gamma;z).\end{aligned}\right\} \tag{77}$$

Now make $z\to 0$. It is readily seen that $C_1=1$. In order to determine the constant C we assume, first, that 2α is one of the numbers $0,-1,-2,\ldots\gamma$. The left side and the first term on the right side of (77) then are polynomials of z of degree -2α, but the second term contains higher powers of z. Therefore C is zero in this case and (77) reduces to

$$\left.\begin{aligned}(1+z)^{-2\alpha}f\left(\alpha,\alpha+\frac{1}{2},\gamma;\frac{4z}{(1+z)^2}\right)&=f(2\alpha,2\alpha-\gamma+1,\gamma;z)\\ &=z^{-2\alpha}f\left(2\alpha,2\alpha-\gamma+1,\gamma;\frac{1}{z}\right),\end{aligned}\right\} \tag{78}$$

provided that 2α has any of the values $0,-1,-2,\ldots\gamma$. The last expression is obtained by reversing the order of the terms.

Next, we assume that $2\alpha=\gamma-1,\gamma-2,\gamma-3,\ldots 2\gamma-1$. The first term on the right side of (77) now is a polynomial of a lower degree than -2α, and the second term is a polynomial which, by reversing the order of the terms, can be written

$$\left.\begin{aligned} & z^{1-\gamma} F(2\alpha-\gamma+1,\, 2\alpha-2\gamma+2,\, 2-\gamma;\, z) = \\ & \frac{(-1)^{\gamma-2\alpha-1}\,\Gamma(1-\gamma)\,\Gamma(2-\gamma)}{\Gamma(2\alpha-2\gamma+2)\,\Gamma(1-2\alpha)}\, z^{-2\alpha} f\left(2\alpha,\, 2\alpha-\gamma+1,\, \gamma;\, \frac{1}{z}\right). \end{aligned}\right\} \quad (79)$$

Now we multiply both sides of (77) by $z^{2\alpha}$ and let $z \to \infty$. We then obtain the relation

$$1 = C\,\frac{(-1)^{\gamma-2\alpha-1}\,\Gamma(1-\gamma)\,\Gamma(2-\gamma)}{\Gamma(2\alpha-2\gamma+2)\,\Gamma(1-2\alpha)},$$

which gives the value of C. Substituting this value in (77), we get

$$(1+z)^{-2\alpha} f\left(\alpha,\, \alpha+\frac{1}{2},\, \gamma;\, \frac{4z}{(1+z)^2}\right) = f(2\alpha,\, 2\alpha-\gamma+1,\, \gamma;\, z) +$$

$$(-1)^{\gamma-2\alpha-1}\frac{\Gamma(2\alpha-2\gamma+2)\,\Gamma(1-2\alpha)}{\Gamma(1-\gamma)\,\Gamma(2-\gamma)}\, z^{1-\gamma} F(2\alpha-\gamma+1,\, 2\alpha-2\gamma+2,\, 2-\gamma;\, z).$$

But using (79) we can also write this relation in the following manner:

$$\left.\begin{aligned} f\left(\alpha,\, \alpha+\frac{1}{2},\, \gamma;\, \frac{4z}{(1+z)^2}\right) &= (1+z)^{2\alpha} f(2\alpha,\, 2\alpha-\gamma+1,\, \gamma;\, z) + \\ & \left(1+\frac{1}{z}\right)^{2\alpha} f\left(2\alpha,\, 2\alpha-\gamma+1,\, \gamma;\, \frac{1}{z}\right), \end{aligned}\right\} \quad (80)$$

where it is supposed that 2α is an integer and $\gamma > 2\alpha \geq 2\gamma - 1$.

§ 12. If we apply the transformation (27) to the left-hand side of (68), we obtain the relation

$$F\left(\alpha,\, \gamma-\alpha-\frac{1}{2},\, \gamma;\, \frac{-4z}{(1-z)^2}\right) = (1-z)^{2\alpha} F(2\alpha,\, 2\alpha-\gamma+1,\, \gamma;\, z), \quad (81)$$

which was given by Kummer.

If γ is a positive integer, it follows from (30) that

$$G\left(\alpha,\, \alpha+\frac{1}{2},\, \gamma;\, \frac{4z}{(1+z)^2}\right) = \left(\frac{1+z}{1-z}\right)^{2\alpha} G\left(\alpha,\, \gamma-\alpha-\frac{1}{2},\, \gamma;\, \frac{-4z}{(1-z)^2}\right) +$$

$$\left(\pm\pi i + \sum_{s=1}^{\gamma-1}\frac{1}{s-\alpha-\frac{1}{2}}\right) F\left(\alpha,\, \alpha+\frac{1}{2},\, \gamma;\, \frac{4z}{(1+z)^2}\right).$$

If we use this relation to transform the left-hand side of (73), we get the following relation

$$\left.\begin{aligned} G\left(\alpha,\, \gamma-\alpha-\frac{1}{2},\, \gamma;\, \frac{-4z}{(1-z)^2}\right) &= (1-z)^{2\alpha} G(2\alpha,\, 2\alpha-\gamma+1,\, \gamma;\, z) \\ & + C_1 F\left(\alpha,\, \gamma-\alpha-\frac{1}{2},\, \gamma;\, \frac{-4z}{(1-z)^2}\right), \end{aligned}\right\} \quad (82)$$

provided that $\alpha \neq \frac{1}{2}, 1, \frac{3}{2}, \ldots \gamma - 1$. The constant C_1 figuring in this relation has the following value

$$C_1 = \log 4 \mp \pi i + \sum_{s=1}^{\gamma-1} \left(\frac{1}{s - 2\alpha} - \frac{1}{s - \alpha - \frac{1}{2}} \right),$$

where the upper or lower sign is to be taken according as $I(z) \gtrless 0$.

If we apply the transformation (37) to the left-hand side of (74), we obtain

$$g_1\left(\gamma - \alpha - \frac{1}{2}, \alpha, \gamma; \frac{-4z}{(1-z)^2}\right) = (1-z)^{2\alpha} g(2\alpha, 2\alpha - \gamma + 1, \gamma; z), \tag{83}$$

provided that 2α is not an integer $< 2\gamma - 1$. Similarly, if we apply the transformation (38) to the left-hand side of (75) we get

$$g_1\left(\alpha, \gamma - \alpha - \frac{1}{2}, \gamma; \frac{-4z}{(1-z)^2}\right) = (1-z)^{2\alpha} g_0(2\alpha, 2\alpha - \gamma + 1, \gamma; z), \tag{84}$$

provided that 2α is not a positive integer. If we use the transformation (35), it follows from (76) that

$$\left.\begin{aligned} g_0\left(\alpha, \gamma - \alpha - \frac{1}{2}, \gamma; \frac{-4z}{(1-z)^2}\right) &= (1-z)^{2\alpha} g_1(2\alpha, 2\alpha - \gamma + 1, \gamma; z) \\ &\quad - \left(1 - \frac{1}{z}\right)^{2\alpha} g_1\left(2\alpha, 2\alpha - \gamma + 1, \gamma; \frac{1}{z}\right) \end{aligned}\right\} \tag{85}$$

under the condition that 2α is not an integer.

If γ is zero or a negative integer, we employ the relation (41) and then from (78) and (80) get the following formulae

$$\left.\begin{aligned} f\left(\alpha, \gamma - \alpha - \frac{1}{2}, \gamma; \frac{-4z}{(1-z)^2}\right) &= (1-z)^{2\alpha} f(2\alpha, 2\alpha - \gamma + 1, \gamma; z) \\ &\quad + \left(1 - \frac{1}{z}\right)^{2\alpha} f\left(2\alpha, 2\alpha - \gamma + 1, \gamma; \frac{1}{z}\right), \end{aligned}\right\} \tag{86}$$

where α has any of the values $\frac{\gamma-1}{2}, \frac{\gamma-2}{2}, \frac{\gamma-3}{2}, \ldots \gamma - \frac{1}{2}$.

$$\left.\begin{aligned} f\left(\alpha, \gamma - \alpha - \frac{1}{2}, \gamma; \frac{-4z}{(1-z)^2}\right) &= (1-z)^{2\alpha} f(2\alpha, 2\alpha - \gamma + 1, \gamma; z) \\ &= \left(1 - \frac{1}{z}\right)^{2\alpha} f\left(2\alpha, 2\alpha - \gamma + 1, \gamma; \frac{1}{z}\right), \end{aligned}\right\} \tag{87}$$

provided that α is a non-positive integer $\geqq \frac{\gamma}{2}$,

$$\left.\begin{aligned} f\left(\alpha, \gamma-\alpha-\frac{1}{2}, \gamma; \frac{-4z}{(1-z)^2}\right) &= (1-z)^{2\gamma-2\alpha-1} f(\gamma-2\alpha, 2\gamma-2\alpha-1, \gamma; z) \\ &+ \left(1-\frac{1}{z}\right)^{2\gamma-2\alpha-1} f\left(\gamma-2\alpha, 2\gamma-2\alpha-1, \gamma; \frac{1}{z}\right), \end{aligned}\right\} \quad (88)$$

provided that 2α is an odd integer and $0 > 2\alpha \geq \gamma$.

§ 13. We now assume that $\beta = \frac{1}{2}\gamma$ instead of $\beta = \alpha + \frac{1}{2}$. The differential equation (66) then reduces to

$$z(1-z^2)\frac{d^2y}{dz^2} + [\gamma - (4\alpha - \gamma + 2) z^2]\frac{dy}{dz} - 2\alpha(2\alpha + 1 - \gamma) zy = 0.$$

Put $z^2 = x$ and we obtain the hypergeometric equation

$$x(1-x)\frac{d^2y}{dx^2} + \left[\frac{\gamma+1}{2} - \left(2\alpha - \frac{\gamma}{2} + \frac{3}{2}\right)x\right]\frac{dy}{dx} - \alpha\left(\alpha + \frac{1-\gamma}{2}\right)y = 0 \quad (89)$$

with the parameters α, $\alpha + \frac{1-\gamma}{2}$ and $\frac{\gamma+1}{2}$. It has the solution $F\left(\alpha, \alpha + \frac{1-\gamma}{2}, \frac{\gamma+1}{2}; z^2\right)$ and (65) has the solution $F\left(\alpha, \frac{\gamma}{2}, \gamma; \frac{4z}{(1+z)^2}\right)$. Hence

$$F\left(\alpha, \frac{\gamma}{2}, \gamma; \frac{4z}{(1+z)^2}\right) = (1+z)^{2\alpha} F\left(\alpha, \alpha + \frac{1-\gamma}{2}, \frac{\gamma+1}{2}; z^2\right). \quad (90)$$

This formula is due to Gauss.

If γ is an odd positive integer and if α is not an integer $< \gamma$, then (65) also has the solution $g\left(\alpha, \frac{\gamma}{2}, \gamma; \frac{4z}{(1+z)^2}\right)$ and (89) also has the solution $g\left(\alpha, \alpha + \frac{1-\gamma}{2}, \frac{\gamma+1}{2}; z^2\right)$. It follows that

$$\begin{aligned} g\left(\alpha, \frac{\gamma}{2}, \gamma; \frac{4z}{(1+z)^2}\right) = (1+z)^{2\alpha}\Bigg[C_1 g\left(\alpha, \alpha + \frac{1-\gamma}{2}, \frac{\gamma+1}{2}; z^2\right) \\ + CF\left(\alpha, \alpha + \frac{1-\gamma}{2}, \frac{\gamma+1}{2}; z^2\right)\Bigg]. \end{aligned}$$

We divide both sides by $z^{1-\gamma}$ (or by $\log z$ if $\gamma = 1$). Let $z \to 0$ and using (60) or (61) we find in both cases that $C_1 = \frac{1}{2}$. Next, let $z \to 1$ and we get $C = 0$. Hence

$$g\left(\alpha, \frac{\gamma}{2}, \gamma; \frac{4z}{(1+z)^2}\right) = \frac{1}{2}(1+z)^{2\alpha} g\left(\alpha, \alpha + \frac{1-\gamma}{2}, \frac{\gamma+1}{2}; z^2\right). \quad (91)$$

If γ is an odd negative integer and if α has any of the values $0, -1, -2, \ldots \gamma$, then (65) has the solution $f\left(\alpha, \frac{\gamma}{2}, \gamma; \frac{4z}{(1+z)^2}\right)$ and (89) has the solutions $f\left(\alpha, \alpha + \frac{1-\gamma}{2}, \frac{\gamma+1}{2}; z^2\right)$ and $z^{1-\gamma} F\left(\alpha + \frac{1-\gamma}{2}, \alpha - \gamma + 1, \frac{3-\gamma}{2}; z^2\right)$. Hence

$$\left.\begin{aligned} (1+z)^{-2\alpha} f\left(\alpha, \frac{\gamma}{2}, \gamma; \frac{4z}{(1+z)^2}\right) &= f\left(\alpha, \alpha + \frac{1-\gamma}{2}, \frac{\gamma+1}{2}; z^2\right) \\ &+ C z^{1-\gamma} F\left(\alpha + \frac{1-\gamma}{2}, \alpha - \gamma + 1, \frac{3-\gamma}{2}; z^2\right). \end{aligned}\right\} \tag{92}$$

In order to determine the constant C we assume, first, that α has one of the values $0, -1, -2, \ldots \frac{\gamma+1}{2}$. The left-hand side and the first term on the right-hand side of (92) then are polynomials of the degree -2α, but the second term contains higher powers of z. Therefore, C is zero in this case. Next, we assume that α has one of the values $\frac{\gamma-1}{2}, \frac{\gamma-3}{2}, \frac{\gamma-5}{2}, \ldots \gamma$. By reversing the order of the terms, we get

$$z^{1-\gamma} F\left(\alpha + \frac{1-\gamma}{2}, \alpha - \gamma + 1, \frac{3-\gamma}{2}; z^2\right) =$$

$$(-1)^{\frac{\gamma-1}{2}-\alpha} \frac{\Gamma\left(\frac{3-\gamma}{2}\right)\Gamma\left(\frac{1-\gamma}{2}\right)}{\Gamma(1-\alpha)\,\Gamma(\alpha-\gamma+1)} z^{-2\alpha} f\left(\alpha, \alpha + \frac{1-\gamma}{2}, \frac{\gamma+1}{2}; \frac{1}{z^2}\right).$$

Now we multiply both sides of (92) by $z^{2\alpha}$ and let $z \to \infty$. We then obtain

$$C = (-1)^{\alpha + \frac{1-\gamma}{2}} \frac{\Gamma(1-\alpha)\,\Gamma(\alpha-\gamma+1)}{\Gamma\left(\frac{1-\gamma}{2}\right)\Gamma\left(\frac{3-\gamma}{2}\right)}.$$

It follows that

$$\begin{aligned} f\left(\alpha, \frac{\gamma}{2}, \gamma; \frac{4z}{(1+z)^2}\right) &= (1+z)^{2\alpha} f\left(\alpha, \alpha + \frac{1-\gamma}{2}, \frac{\gamma+1}{2}; z^2\right) \\ &+ \left(1 + \frac{1}{z}\right)^{2\alpha} f\left(\alpha, \alpha + \frac{1-\gamma}{2}, \frac{\gamma+1}{2}; z^{-2}\right), \end{aligned}$$

provided that $\alpha = \frac{\gamma-1}{2}, \frac{\gamma-3}{2}, \frac{\gamma-5}{2}, \ldots \gamma$, and

$$\begin{aligned} f\left(\alpha, \frac{\gamma}{2}, \gamma; \frac{4z}{(1+z)^2}\right) &= (1+z)^{2\alpha} f\left(\alpha, \alpha + \frac{1-\gamma}{2}, \frac{\gamma+1}{2}; z^2\right) \\ &= \left(1 + \frac{1}{z}\right)^{2\alpha} f\left(\alpha, \alpha + \frac{1-\gamma}{2}, \frac{\gamma+1}{2}; z^{-2}\right), \end{aligned}$$

provided that $\alpha = 0, -1, -2, \ldots \frac{\gamma+1}{2}$.

§ 14. If we put $z = \frac{1-\sqrt{1-x}}{2}$, the differential equation (65) is transformed to

$$z(1-z)\frac{d^2Y}{dz^2} + [\gamma - (4\alpha + 4\beta + 2)z + (4\alpha + 4\beta + 2)z^2]\frac{1}{1-2z}\frac{dY}{dz} - 4\alpha\beta Y = 0.$$

If $\gamma = \alpha + \beta + \frac{1}{2}$, this equation takes the form

$$z(1-z)\frac{d^2Y}{dz^2} + \left[\alpha + \beta + \frac{1}{2} - (2\alpha + 2\beta + 1)z\right]\frac{dY}{dz} - 4\alpha\beta Y = 0. \tag{93}$$

It has the solution $F\left(2\alpha, 2\beta, \alpha+\beta+\frac{1}{2}; \frac{1-\sqrt{1-x}}{2}\right)$. Hence, we have the relation

$$F\left(2\alpha, 2\beta, \alpha+\beta+\frac{1}{2}; \frac{1-\sqrt{1-x}}{2}\right) = F\left(\alpha, \beta, \alpha+\beta+\frac{1}{2}; x\right), \tag{94}$$

a result given by Gauss.

If γ is a positive integer, and if 2α is not an integer, then (93) has the second solution $g\left(2\alpha, 2\beta, \alpha+\beta+\frac{1}{2}; \frac{1-\sqrt{1-x}}{2}\right)$, and $g\left(\alpha, \beta, \alpha+\beta+\frac{1}{2}; x\right)$ is a solution of (65). It follows that there exist constants C and C_1 such that

$$g\left(2\alpha, 2\beta, \alpha+\beta+\frac{1}{2}; \frac{1-\sqrt{1-x}}{2}\right) = C_1 g\left(\alpha, \beta, \alpha+\beta+\frac{1}{2}; x\right) + CF\left(\alpha, \beta, \alpha+\beta+\frac{1}{2}; x\right).$$

Now make $x \to 0$. It is readily seen that $C_1 = 1$. Next, we let $x \to 1$ and we obtain

$$\left.\begin{aligned} C\frac{\Gamma\left(\alpha+\beta+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\beta+\frac{1}{2}\right)} &= g\left(2\alpha, 2\beta, \alpha+\beta+\frac{1}{2}; \frac{1}{2}\right) \\ &\quad - g\left(\alpha, \beta, \alpha+\beta+\frac{1}{2}; 1\right). \end{aligned}\right\} \tag{95}$$

Using (46) we see that

$$g\left(\alpha, \beta, \alpha+\beta+\frac{1}{2}; 1\right) = (-1)^{\alpha+\beta+\frac{1}{2}}\frac{\Gamma\left(\alpha+\beta+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}-\alpha\right)\Gamma\left(\frac{1}{2}-\beta\right)}{\Gamma\left(\frac{1}{2}\right)},$$

$$\left.\begin{aligned} & g\left(2\alpha, 2\beta, \alpha+\beta+\frac{1}{2}; \frac{1}{2}\right) \\ & = (-1)^{\alpha+\beta+\frac{1}{2}} \Gamma\left(\alpha-\beta+\frac{1}{2}\right) \Gamma\left(\beta-\alpha+\frac{1}{2}\right) F\left(2\alpha, 2\beta, \alpha+\beta+\frac{1}{2}; \frac{1}{2}\right) \\ & = (-1)^{\alpha+\beta+\frac{1}{2}} \frac{\Gamma\left(\alpha+\beta+\frac{1}{2}\right) \Gamma\left(\alpha-\beta+\frac{1}{2}\right) \Gamma\left(\beta-\alpha+\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\alpha+\frac{1}{2}\right) \Gamma\left(\beta+\frac{1}{2}\right)}. \end{aligned}\right\} \quad (96)$$

Substituting these values in (95), we get

$$C = \frac{\pi}{\sin 2\pi\alpha}.$$

It follows that

$$\left.\begin{aligned} g\left(2\alpha, 2\beta, \alpha+\beta+\frac{1}{2}; \frac{1-\sqrt{1-x}}{2}\right) &= g\left(\alpha, \beta, \alpha+\beta+\frac{1}{2}; x\right) \\ &+ \frac{\pi}{\sin 2\pi\alpha} F\left(\alpha, \beta, \alpha+\beta+\frac{1}{2}; x\right), \end{aligned}\right\} \quad (97)$$

provided that 2α is not an integer. Using (20) we derive from this formula that

$$\left.\begin{aligned} G\left(2\alpha, 2\beta, \alpha+\beta+\frac{1}{2}; \frac{1-\sqrt{1-x}}{2}\right) &= G\left(\alpha, \beta, \alpha+\beta+\frac{1}{2}; x\right) \\ &+ K F\left(\alpha, \beta, \alpha+\beta+\frac{1}{2}; x\right), \end{aligned}\right\} \quad (98)$$

where the constant K has the value

$$K = \Psi(\alpha) + \Psi(\beta) - \Psi(2\alpha) - \Psi(2\beta) + \frac{\pi}{\sin 2\pi\alpha},$$

and it is easily seen that this expression reduces to

$$\begin{aligned} K &= \sum_{s=1}^{2\gamma-2} \frac{(-1)^s}{2\alpha-s} - \log 4 \\ &= \sum_{s=1}^{2\gamma-2} \frac{(-1)^s}{2\beta-s} - \log 4. \end{aligned}$$

The relation (98) is true if $2\alpha \neq 1, 2, 3, \ldots 2\gamma-2$.
In a similar way it is seen that

$$g\left(2\alpha, 2\beta, \alpha+\beta+\frac{1}{2}; \frac{1+\sqrt{1-x}}{2}\right) = -\frac{\pi}{\sin 2\pi\alpha} F\left(\alpha, \beta, \alpha+\beta+\frac{1}{2}; x\right).$$

If $\gamma = \alpha+\beta+\frac{1}{2}$ is a negative integer, $f\left(2\alpha, 2\beta, \gamma; \frac{1-\sqrt{1-x}}{2}\right)$ is a solution of (93), again $f(\alpha, \beta, \gamma; x)$ and $x^{1-\gamma} F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma; x)$ are solutions of (65), provided that α has any of the values $0, -\frac{1}{2}, -1, -\frac{3}{2}, \ldots \gamma-\frac{1}{2}$. These solutions are connected by a linear relation

$$\left.\begin{aligned} f\left(2\alpha, 2\beta, \gamma; \frac{1-\sqrt{1-x}}{2}\right) = f(\alpha, \beta, \gamma; x) + \\ Cx^{1-\gamma} F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma; x), \end{aligned}\right\} \tag{99}$$

C being a constant. If $\alpha = 0, -1, -2, \ldots -\left[\frac{|\gamma|}{2}\right]$ we have $\beta < \alpha$. For large negative values of x the left-hand side and the first term on the right-hand side of (99) are $O(|x|^{-\alpha})$, while the second term on the right is $O(|x|^{-\beta})$. But if $\alpha = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots \frac{1}{2}+\left[\frac{\gamma}{2}\right]$ the left side is again $O(|x|^{-\alpha})$, whereas both terms on the right side are $O(|x|^{-\beta})$. Multiply both sides of (99) by $(-x)^{\beta}$ and let $x \to -\infty$. In the first case we get $C = 0$. Using Table 11 Formulae (1) and (7), we get in the second case

$$C = -\frac{\Gamma\left(\frac{1}{2}-\alpha\right)\Gamma\left(\frac{1}{2}-\beta\right)\Gamma(1-\alpha)\Gamma(1-\beta)}{\pi\,\Gamma(1-\gamma)\Gamma(2-\gamma)}. \tag{100}$$

Now in the second case (99) can be reduced. If γ is a negative integer and if β has one of the values $0, -1, -2, \ldots \gamma$, it follows from (44) that

$$\begin{aligned} (1-x)^{\gamma-\alpha-\beta} f(\gamma-\alpha, \gamma-\beta, \gamma; x) = f(\alpha, \beta, \gamma; x) + \\ Cx^{1-\gamma} F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma; x), \end{aligned}$$

where C has the value (100). As in the actual case $\gamma-\alpha-\beta = \frac{1}{2}$ Formula (99) reduces to

$$f\left(2\alpha, 2\beta, \gamma; \frac{1-\sqrt{1-x}}{2}\right) = \sqrt{1-x}\, f\left(\alpha+\frac{1}{2}, \beta+\frac{1}{2}, \gamma; x\right), \tag{101}$$

provided that α or β is equal to one of the numbers $-\frac{1}{2}, -\frac{3}{2}, \ldots \frac{1}{2}+\left[\frac{\gamma}{2}\right]$. Furthermore

$$f\left(2\alpha, 2\beta, \gamma; \frac{1-\sqrt{1-x}}{2}\right) = f(\alpha, \beta, \gamma; x), \tag{102}$$

provided that α or β is equal to one of the numbers $0, -1, -2, \ldots -\left[\frac{|\gamma|}{2}\right]$. Finally, we have two expansions proceeding in descending powers of x

$$f\left(2\alpha, 2\beta, \gamma; \frac{1-\sqrt{1-x}}{2}\right) = \frac{\Gamma\left(\frac{1}{2}-\beta\right)\Gamma(1-\beta)}{\Gamma(1-\gamma)\Gamma(\alpha-\beta+1)}(-x)^{-\alpha} F\left(\alpha, \frac{1}{2}-\beta, \alpha-\beta+1; \frac{1}{x}\right),$$

when $2\alpha = 0, -1, -2, \ldots \gamma$ and

$$f\left(2\alpha, 2\beta, \gamma; \frac{1-\sqrt{1-x}}{2}\right) = \frac{\Gamma\left(\frac{1}{2}-\alpha\right)\Gamma(1-\alpha)}{\Gamma(1-\gamma)\Gamma(\beta-\alpha+1)}(-x)^{-\beta} F\left(\frac{1}{2}-\alpha, \beta, \beta-\alpha+1; \frac{1}{x}\right),$$

when $2\beta = 0, -1, -2, \ldots \gamma$.

§ 15. Some special cases may be set down. Putting $x = 1$ in (94), we get

$$F\left(2\alpha, 2\beta, \alpha+\beta+\frac{1}{2}; \frac{1}{2}\right) = \frac{\Gamma\left(\alpha+\beta+\frac{1}{2}\right)\sqrt{\pi}}{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\beta+\frac{1}{2}\right)}. \tag{103}$$

Putting $z = -1$ in (81), we obtain

$$F(2\alpha, 2\alpha-\gamma+1, \gamma; -1) = \frac{2^{-2\alpha}\Gamma(\gamma)\sqrt{\pi}}{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma(\gamma-\alpha)}, \quad \gamma \neq 0, -1, -2, \ldots \tag{104}$$

The first of these formulae is due to Gauss and the second is due to Kummer. If we apply the transformation (28) on the left side of Gauss' formula we obtain

$$F\left(\alpha, 1-\alpha, \gamma; \frac{1}{2}\right) = \frac{2^{1-\gamma}\Gamma(\gamma)\sqrt{\pi}}{\Gamma\left(\frac{\alpha+\gamma}{2}\right)\Gamma\left(\frac{\gamma-\alpha+1}{2}\right)}, \quad \gamma \neq 0, -1, -2, \ldots \tag{105}$$

If γ is a negative integer and either α or β is equal to one of the numbers $0, -1, -2, \ldots \gamma$. we have for the rational solution as shown in § 8

$$f(\alpha, \beta, \gamma; 1) = \frac{\Gamma(\alpha-\gamma+1)\Gamma(\beta-\gamma+1)}{\Gamma(1-\gamma)\Gamma(\alpha+\beta-\gamma+1)}.$$

If $\gamma = \alpha+\beta+\frac{1}{2}$, we get from (102) and (101)

$$f\left(2\alpha, 2\beta, \gamma; \frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}-\alpha\right)\Gamma\left(\frac{1}{2}-\beta\right)}{\Gamma(1-\gamma)\Gamma\left(\frac{1}{2}\right)},$$

when α or $\beta = 0, -1, -2, \ldots -\left[\frac{|\gamma|}{2}\right]$, and

$$f\left(2\alpha, 2\beta, \gamma; \frac{1}{2}\right) = 0,$$

when α or $\beta = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots \frac{1}{2} + \left[\frac{\gamma}{2}\right]$. Putting $z = -1$ in (87) and (86), we obtain

$$f(2\alpha, 2\alpha - \gamma + 1, \gamma; -1) = \frac{\Gamma(\alpha - \gamma + 1)\,\Gamma\left(\frac{1}{2} - \alpha\right)}{\Gamma(1-\gamma)\,\Gamma\left(\frac{1}{2}\right) 2^{2\alpha}},$$

when α is a non-positive integer $> \frac{\gamma}{2}$, and

$$f(2\alpha, 2\alpha - \gamma + 1, \gamma; -1) = \frac{\Gamma(\alpha - \gamma + 1)\,\Gamma\left(\frac{1}{2} - \alpha\right)}{\Gamma(1-\gamma)\,\Gamma\left(\frac{1}{2}\right) 2^{2\alpha+1}},$$

when α has any of the values $\frac{\gamma-1}{2}, \frac{\gamma-2}{2}, \frac{\gamma-3}{2}, \ldots \gamma - \frac{1}{2}$. Applying the transformation (43), we get from the last-mentioned formula

$$f\left(\alpha, 1-\alpha, \gamma; \frac{1}{2}\right) = \frac{\Gamma\left(1 - \frac{\alpha+\gamma}{2}\right)\Gamma\left(\frac{\alpha-\gamma+1}{2}\right)}{\Gamma(1-\gamma)\,\Gamma\left(\frac{1}{2}\right) 2^{\gamma}},$$

provided that α is an integer and $\gamma < \alpha < 1 - \gamma$, γ as before being a negative integer.

In the following formulae for the logarithmic solutions γ is a positive integer. If $\gamma = \alpha + \beta + \frac{1}{2}$, we get from (96)

$$g\left(2\alpha, 2\beta, \gamma; \frac{1}{2}\right) = \frac{(-1)^{\gamma}}{2\sqrt{\pi}}\,\Gamma(\gamma)\,\Gamma\left(\frac{1}{2} - \alpha\right)\Gamma\left(\frac{1}{2} - \beta\right),$$

provided that $2\alpha \neq 0, \pm 1, \pm 2, \ldots$ Putting $x = 1$ in (98), we obtain

$$G\left(2\alpha, 2\beta, \gamma; \frac{1}{2}\right) = \frac{\Gamma(\gamma)\sqrt{\pi}}{\Gamma\left(\alpha + \frac{1}{2}\right)\Gamma\left(\beta + \frac{1}{2}\right)}\left[\Psi(\gamma) + \Psi(1) - \Psi\left(\alpha + \frac{1}{2}\right)\right.$$

$$\left. - \Psi\left(\beta + \frac{1}{2}\right) - \log 4 - \sum_{s=1}^{2\gamma-2} \frac{(-1)^s}{2\alpha - s}\right],$$

provided that $2\alpha \neq 1, 2, 3, \ldots 2\gamma - 2$.

If we let $z \to -1$ in (82), we get

$$2^{2\alpha} G(2\alpha, 2\alpha-\gamma+1, \gamma; -1 \pm io) = G\left(\alpha, \gamma-\alpha-\frac{1}{2}, \gamma; 1\right) - C_1 F\left(\alpha, \gamma-\alpha-\frac{1}{2}, \gamma; 1\right).$$

Using (54) we find that

$$\left.\begin{aligned} G(2\alpha, 2\alpha-\gamma+1, \gamma; -1 \pm io) = \frac{\Gamma(\gamma)\sqrt{\pi}\, 2^{-2\alpha}}{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma(\gamma-\alpha)} \Big[\Psi(\gamma)+\Psi(1)-\Psi\left(\alpha+\frac{1}{2}\right) \\ -\Psi(1-\alpha)-\log 4 \pm \pi i + \sum_{s=1}^{\gamma-1} \frac{1}{2\alpha-s}\Big], \end{aligned}\right\} \quad (106)$$

if $2\alpha \neq 1, 2, 3, \ldots 2\gamma-2$. Substituting this value in (20) and using (104), we obtain, after some manipulations

$$g(2\alpha, 2\alpha-\gamma+1, \gamma; -1 \pm io) = (-1)^{\gamma} \frac{\Gamma(\gamma)\Gamma(\alpha-\gamma+1)\sqrt{\pi}}{\Gamma\left(\alpha+\frac{1}{2}\right) 2^{2\alpha}} e^{\pm \pi i \alpha},$$

provided that 2α is not an integer $< 2\gamma-1$. This result could also be obtained from (83). At the point -1 the difference of the values on the opposite edges of the cross-cut from 0 to $-\infty$ is thus

$$\begin{aligned} G(2\alpha, 2\alpha-\gamma+1, \gamma; -1+io) - G(2\alpha, 2\alpha-\gamma+1, \gamma; -1-io) = \\ g(2\alpha, 2\alpha-\gamma+1, \gamma; -1+io) - g(2\alpha, 2\alpha-\gamma+1, \gamma; -1-io) = \\ \frac{\Gamma(\gamma)\sqrt{\pi}\, 2^{-2\alpha}\, 2\pi i}{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma(\gamma-\alpha)}. \end{aligned}$$

Using (30), (104), and (106) we find that

$$G\left(\alpha, 1-\alpha, \gamma; \frac{1}{2}\right) = \frac{\Gamma(\gamma)\sqrt{\pi}\, 2^{1-\gamma}}{\Gamma\left(\frac{\alpha+\gamma}{2}\right)\Gamma\left(\frac{\gamma-\alpha+1}{2}\right)} \left[\Psi(\gamma)+\Psi(1)-2\Psi(\alpha)+\pi \operatorname{tg}\pi\left(\frac{\alpha+\gamma}{2}\right)\right],$$

provided that $\alpha \neq 0, \pm 1, \pm 2, \ldots \pm(\gamma-2), \gamma-1$. From this formula we get, using (20),

$$g\left(\alpha, 1-\alpha, \gamma; \frac{1}{2}\right) = \frac{(-1)^{\gamma}}{\sqrt{\pi}} \Gamma(\gamma)\Gamma\left(1-\frac{\alpha+\gamma}{2}\right)\Gamma\left(\frac{\alpha-\gamma+1}{2}\right) 2^{-\gamma},$$

where α is not an integer. Again from (24), using (105), we obtain

$$g_1\left(\alpha,\,1-\alpha,\,\gamma;\,\frac{1}{2}\pm io\right)=\frac{\Gamma(\gamma)\sqrt{\pi}\,\Gamma\left(1-\frac{\alpha+\gamma}{2}\right)}{2^{\gamma-1}\,\Gamma\left(\frac{\gamma-\alpha+1}{2}\right)}\,e^{\mp\frac{\pi i\,(\alpha+\gamma)}{2}},$$

under the condition that $\alpha+\gamma-1$ is not a positive integer. It follows that

$$g_1\left(\alpha,\,1-\alpha,\,\gamma;\,\frac{1}{2}+io\right)-g_1\left(\alpha,\,1-\alpha,\,\gamma;\,\frac{1}{2}-io\right)=-\frac{\Gamma(\gamma)\sqrt{\pi}\,2^{1-\gamma}\,2\pi i}{\Gamma\left(\frac{\alpha+\gamma}{2}\right)\Gamma\left(\frac{\gamma-\alpha+1}{2}\right)}.$$

Chapter III

Riemann's Differential Equation

§ 16. We now consider the exceptional solutions of the differential equation for Riemann's P-function

$$\frac{d^2y}{dz^2}+\left(\frac{1-\alpha-\alpha'}{z}-\frac{1-\gamma-\gamma'}{1-z}\right)\frac{dy}{dz}+\left(\frac{\alpha\alpha'}{z}+\frac{\gamma\gamma'}{1-z}-\beta\beta'\right)\frac{y}{z(1-z)}=0, \tag{1}$$

where $\alpha+\alpha'+\beta+\beta'+\gamma+\gamma'=1$. The interrelations of the fundamental solutions can of course in this case be written in a shorter form. If $\alpha-\alpha'$ is not an integer, (1) admits near $z=0$ the linearly independent solutions

$$P^{\alpha}=z^{\alpha}(1-z)^{\gamma}F(\alpha+\beta+\gamma,\,\alpha+\beta'+\gamma,\,\alpha-\alpha'+1;\,z),$$
$$P^{\alpha'}=z^{\alpha'}(1-z)^{\gamma}F(\alpha'+\beta+\gamma,\,\alpha'+\beta'+\gamma,\,\alpha'-\alpha+1;\,z).$$

If $\gamma-\gamma'$ is not an integer, we have in the vicinity of $z=1$ the solutions

$$P^{\gamma}=(1-z)^{\gamma}z^{\alpha}F(\alpha+\beta+\gamma,\,\alpha+\beta'+\gamma,\,\gamma-\gamma'+1;\,1-z),$$
$$P^{\gamma'}=(1-z)^{\gamma'}z^{\alpha}F(\alpha+\beta+\gamma',\,\alpha+\beta'+\gamma',\,\gamma'-\gamma+1;\,1-z).$$

If $\beta-\beta'$ is non-integer, (1) has for $|z|>1$ the solutions

$$P^{\beta}=z^{-\beta}\left(1-\frac{1}{z}\right)^{\gamma}F\left(\alpha+\beta+\gamma,\,\alpha'+\beta+\gamma,\,\beta-\beta'+1;\,\frac{1}{z}\right),$$

$$P^{\beta'}=z^{-\beta'}\left(1-\frac{1}{z}\right)^{\gamma}F\left(\alpha+\beta'+\gamma,\,\alpha'+\beta'+\gamma,\,\beta'-\beta+1;\,\frac{1}{z}\right).$$

We now put

$$Q^{\alpha}_{\gamma}=z^{\alpha}(1-z)^{\gamma}g(\alpha+\beta+\gamma,\,\alpha+\beta'+\gamma,\,\alpha-\alpha'+1;\,z), \tag{2}$$

$$p^{\alpha}_{\gamma}=z^{\alpha}(1-z)^{\gamma}f(\alpha+\beta+\gamma,\,\alpha+\beta'+\gamma,\,\alpha-\alpha'+1;\,z). \tag{3}$$

If we interchange γ and γ' in (2) and (3), we get $Q^{\alpha}_{\gamma'}$ and $p^{\alpha}_{\gamma'}$. If we interchange in (2) and (3) α and γ, z and $1-z$, we get Q^{γ}_{α} and p^{γ}_{α}. Finally, interchanging α and β, z and $\frac{1}{z}$ in (2) and (3), we get Q^{β}_{γ} and p^{β}_{γ}.

In a similar way we put

$$R^{\alpha}_{\beta} = z^{\alpha}(1-z)^{\gamma}\, g_1(\alpha+\beta+\gamma,\ \alpha+\beta'+\gamma,\ \alpha-\alpha'+1\,;\ z). \tag{4}$$

If we interchange β and β' in (4), we get $R^{\alpha}_{\beta'}$. If we interchange in (4) α and γ, z and $1-z$, we get R^{γ}_{β}. Finally, interchanging α and β, z and $\frac{1}{z}$ in (4) we get R^{β}_{α}. For brevity's sake we shall write

$$\alpha_{\gamma} = \frac{\Gamma(\alpha-\alpha'+1)\,\Gamma(\gamma'-\gamma)}{\Gamma(\alpha+\beta+\gamma')\,\Gamma(\alpha+\beta'+\gamma')},$$

$$\dot{\alpha}_{\gamma} = (-1)^{\alpha-\alpha'+1}\,\frac{\Gamma(\alpha'+\beta+\gamma)\,\Gamma(\alpha'+\beta'+\gamma)\,\Gamma(\alpha-\alpha'+1)}{\Gamma(\gamma-\gamma'+1)}.$$

The other constants follow from these by interchange of letters, for example

$$\gamma_{\alpha} = \frac{\Gamma(\gamma-\gamma'+1)\,\Gamma(\alpha'-\alpha)}{\Gamma(\alpha'+\beta+\gamma)\,\Gamma(\alpha'+\beta'+\gamma)},$$

$$\dot{\gamma}_{\alpha} = (-1)^{\gamma-\gamma'+1}\,\frac{\Gamma(\alpha+\beta+\gamma')\,\Gamma(\alpha+\beta'+\gamma')\,\Gamma(\gamma-\gamma'+1)}{\Gamma(\alpha-\alpha'+1)}.$$

If $\alpha-\alpha'$ equals a positive integer p, (1) admits the solutions P^{α} and Q^{α}_{γ}, provided that $\alpha+\beta+\gamma$ and $\alpha+\beta'+\gamma \neq p,\, p-1,\, p-2, \ldots$ But if $\alpha+\beta+\gamma$ or $\alpha+\beta'+\gamma = 1, 2, \ldots p$, we have the linearly independent solutions P^{α} and $p^{\alpha'}_{\gamma}$. Similarly for the other cases.

The continuation formulae now take the following form

1. $\gamma-\gamma' = q$, $\alpha-\alpha'$ not an integer.

$$\dot{\gamma}_{\alpha}\,P^{\alpha} = Q^{\gamma}_{\alpha}, \quad \alpha+\beta+\gamma \text{ and } \alpha+\beta'+\gamma \neq q,\, q-1,\, q-2, \ldots \tag{1}$$

$$\dot{\gamma}_{\alpha'}\,P^{\alpha'} = Q^{\gamma}_{\alpha'}, \quad \alpha+\beta+\gamma \text{ and } \alpha+\beta'+\gamma \neq 1, 2, 3, \ldots \tag{2}$$

$$\gamma_{\alpha}\,P^{\alpha} = P^{\gamma}, \quad \alpha+\beta+\gamma \text{ or } \alpha+\beta'+\gamma = 0, -1, -2, \ldots \tag{3}$$

$$\gamma_{\alpha'}\,P^{\alpha'} = P^{\gamma}, \quad \alpha+\beta+\gamma \text{ or } \alpha+\beta'+\gamma = q+1,\, q+2,\, q+3, \ldots \tag{4}$$

$$P^{\alpha} = \alpha_{\gamma'}\,p^{\gamma'}_{\alpha}, \quad \alpha+\beta+\gamma \text{ or } \alpha+\beta'+\gamma = 1, 2, \ldots q \tag{5}$$

$$P^{\alpha'} = \alpha'_{\gamma'}\,p^{\gamma'}_{\alpha'}, \quad \alpha+\beta+\gamma \text{ or } \alpha+\beta'+\gamma = 1, 2, \ldots q. \tag{6}$$

2. $\alpha - \alpha' = p, \quad \gamma - \gamma'$ not an integer.

$$P^{\alpha} = \alpha_{\gamma}\, P^{\gamma} + \alpha_{\gamma'}\, P^{\gamma'}, \tag{1}$$

$$Q^{\alpha}_{\gamma} = \dot{\alpha}_{\gamma}\, P^{\gamma}, \quad \alpha+\beta+\gamma \quad \text{and} \quad \alpha+\beta'+\gamma \neq p,\, p-1,\, p-2, \ldots \tag{2}$$

$$Q^{\alpha}_{\gamma'} = \dot{\alpha}_{\gamma'}\, P^{\gamma'}, \quad \alpha+\beta+\gamma \quad \text{and} \quad \alpha+\beta'+\gamma \neq 1, 2, 3, \ldots \tag{3}$$

$$P^{\alpha} = \alpha_{\gamma}\, P^{\gamma}, \quad \alpha+\beta+\gamma \quad \text{or} \quad \alpha+\beta'+\gamma = 0, -1, -2, \ldots \tag{4}$$

$$P^{\alpha} = \alpha_{\gamma'}\, P^{\gamma'}, \quad \alpha+\beta+\gamma \quad \text{or} \quad \alpha+\beta'+\gamma = p+1,\, p+2,\, p+3, \ldots \tag{5}$$

$$\gamma_{\alpha'}\, p^{\alpha'}_{\gamma} = P^{\gamma}, \quad \alpha+\beta+\gamma \quad \text{or} \quad \alpha+\beta'+\gamma = 1, 2, \ldots p \tag{6}$$

$$\gamma'_{\alpha'}\, p^{\alpha'}_{\gamma'} = P^{\gamma'}, \quad \alpha+\beta+\gamma \quad \text{or} \quad \alpha+\beta'+\gamma = 1, 2, \ldots p. \tag{7}$$

3. $\alpha - \alpha' = p, \quad \gamma - \gamma' = q.$

$$\dot{\gamma}_{\alpha}\, P^{\alpha} = Q^{\gamma}_{\alpha}, \quad \alpha+\beta+\gamma \quad \text{and} \quad \alpha+\beta'+\gamma \neq q,\, q-1,\, q-2, \ldots \tag{1}$$

$$Q^{\alpha}_{\gamma} = \dot{\alpha}_{\gamma}\, P^{\gamma}, \quad \alpha+\beta+\gamma \quad \text{and} \quad \alpha+\beta'+\gamma \neq p,\, p-1,\, p-2, \ldots \tag{2}$$

$$R^{\alpha}_{\beta} = e^{\mp \pi i\, (\alpha+\beta+\gamma)}\, \frac{\sin \pi\, (\alpha+\beta+\gamma)}{\pi}\, \dot{\alpha}_{\gamma}\, R^{\gamma}_{\beta}, \quad \alpha+\beta+\gamma \neq 1, 2, 3, \ldots \tag{3}$$

$$R^{\alpha}_{\beta'} = e^{\mp \pi i\, (\alpha+\beta'+\gamma)}\, \frac{\sin \pi\, (\alpha+\beta'+\gamma)}{\pi}\, \dot{\alpha}_{\gamma}\, R^{\gamma}_{\beta'}, \quad \alpha+\beta'+\gamma \neq 1, 2, 3, \ldots \tag{4}$$

$$P^{\alpha} = (-1)^n \frac{(q+1)_n}{(p+1)_n}\, P^{\gamma}, \quad \alpha+\beta+\gamma \quad \text{or} \quad \alpha+\beta'+\gamma = -n,\; n = 0, 1, 2, \ldots \tag{5}$$

$$P^{\alpha} = \alpha_{\gamma'}\, p^{\gamma'}_{\alpha} \quad \alpha+\beta+\gamma \quad \text{or} \quad \alpha+\beta'+\gamma = 1, 2, \ldots q \tag{6}$$

$$\gamma_{\alpha'}\, p^{\alpha'}_{\gamma} = P^{\gamma}, \quad \alpha+\beta+\gamma \quad \text{or} \quad \alpha+\beta'+\gamma = 1, 2, \ldots p. \tag{7}$$

4. $\beta - \beta' = m, \quad \alpha - \alpha'$ not an integer.

$$\dot{\beta}_{\alpha}\, P^{\alpha} = e^{\pm \pi i\, (\alpha+\beta)}\, R^{\beta}_{\alpha'}, \quad \alpha+\beta+\gamma \quad \text{and} \quad \alpha+\beta+\gamma' \neq m,\, m-1,\, m-2, \ldots \tag{1}$$

$$\dot{\beta}_{\alpha'}\, P^{\alpha'} = e^{\pm \pi i\, (\alpha'+\beta)}\, R^{\beta}_{\alpha}, \quad \alpha+\beta+\gamma \quad \text{and} \quad \alpha+\beta+\gamma' \neq 1, 2, 3, \ldots \tag{2}$$

$$\beta_{\alpha}\, P^{\alpha} = e^{\pm \pi i\, (\alpha+\beta)}\, P^{\beta}, \quad \alpha+\beta+\gamma \quad \text{or} \quad \alpha+\beta+\gamma' = 0, -1, -2, \ldots \tag{3}$$

$$\beta_{\alpha'}\, P^{\alpha'} = e^{\pm \pi i\, (\alpha'+\beta)}\, P^{\beta}, \quad \alpha+\beta+\gamma \quad \text{or} \quad \alpha+\beta+\gamma' = m+1,\, m+2,\, m+3, \ldots \tag{4}$$

$$P^{\alpha} = e^{\pm \pi i (\alpha + \beta')} \, \alpha_{\beta'} \, p^{\beta'}_{\gamma}, \qquad \alpha + \beta + \gamma = 1, 2, \ldots m \tag{5}$$

$$P^{\alpha} = e^{\pm \pi i (\alpha + \beta')} \, \alpha_{\beta'} \, p^{\beta'}_{\gamma'}, \qquad \alpha + \beta + \gamma' = 1, 2, \ldots m \tag{6}$$

$$P^{\alpha'} = e^{\pm \pi i (\alpha' + \beta')} \, \alpha'_{\beta'} \, p^{\beta'}_{\gamma}, \qquad \alpha + \beta + \gamma' = 1, 2, \ldots m \tag{7}$$

$$P^{\alpha'} = e^{\pm \pi i (\alpha' + \beta')} \, \alpha'_{\beta'} \, p^{\beta'}_{\gamma'}, \qquad \alpha + \beta + \gamma = 1, 2, \ldots m. \tag{8}$$

5. $\alpha - \alpha' = p, \quad \beta - \beta'$ not an integer.

$$P^{\alpha} = e^{\pm \pi i (\alpha + \beta)} \, \alpha_{\beta} \, P^{\beta} + e^{\pm \pi i (\alpha + \beta')} \, \alpha_{\beta'} \, P^{\beta'}, \tag{1}$$

$$R^{\alpha}_{\beta} = e^{\pm \pi i (\alpha + \beta')} \, \dot{\alpha}_{\beta'} \, P^{\beta'}, \qquad \alpha + \beta + \gamma \quad \text{and} \quad \alpha + \beta + \gamma' \neq 1, 2, 3, \ldots \tag{2}$$

$$R^{\alpha}_{\beta'} = e^{\pm \pi i (\alpha + \beta)} \, \dot{\alpha}_{\beta} \, P^{\beta}, \qquad \alpha + \beta + \gamma \quad \text{and} \quad \alpha + \beta + \gamma' \neq p, p-1, p-2, \ldots \tag{3}$$

$$P^{\alpha} = e^{\pm \pi i (\alpha + \beta)} \, \alpha_{\beta} \, P^{\beta}, \qquad \alpha + \beta + \gamma \quad \text{or} \quad \alpha + \beta + \gamma' = 0, -1, -2, \ldots \tag{4}$$

$$P^{\alpha} = e^{\pm \pi i (\alpha + \beta')} \, \alpha_{\beta'} \, P^{\beta'}, \qquad \alpha + \beta + \gamma \quad \text{or} \quad \alpha + \beta + \gamma' = p+1, p+2, p+3, \ldots \tag{5}$$

$$\beta_{\alpha'} \, p^{\alpha'}_{\gamma} = e^{\pm \pi i (\alpha' + \beta)} \, P^{\beta}, \qquad \alpha + \beta + \gamma = 1, 2, \ldots p \tag{6}$$

$$\beta'_{\alpha'} \, p^{\alpha'}_{\gamma} = e^{\pm \pi i (\alpha' + \beta')} \, P^{\beta'}, \qquad \alpha + \beta + \gamma' = 1, 2, \ldots p \tag{7}$$

$$\beta'_{\alpha'} \, p^{\alpha'}_{\gamma'} = e^{\pm \pi i (\alpha' + \beta')} \, P^{\beta'}, \qquad \alpha + \beta + \gamma = 1, 2, \ldots p \tag{8}$$

$$\beta_{\alpha'} \, p^{\alpha'}_{\gamma'} = e^{\pm \pi i (\alpha' + \beta)} \, P^{\beta}, \qquad \alpha + \beta + \gamma' = 1, 2, \ldots p. \tag{9}$$

6. $\alpha - \alpha' = p, \quad \beta - \beta' = m.$

$$\beta_{\alpha} \, P^{\alpha} = e^{\pm \pi i (\alpha + \beta)} \, R^{\beta}_{\alpha'}, \qquad \alpha + \beta + \gamma \quad \text{and} \quad \alpha + \beta + \gamma' \neq m, m-1, m-2, \ldots \tag{1}$$

$$R^{\alpha}_{\beta'} = e^{\pm \pi i (\alpha + \beta)} \, \dot{\alpha}_{\beta} \, P^{\beta}, \qquad \alpha + \beta + \gamma \quad \text{and} \quad \alpha + \beta + \gamma' \neq p, \; p-1, \; p-2, \ldots \tag{2}$$

$$Q^{\alpha}_{\gamma} = -\frac{e^{\mp \pi i \gamma}}{\pi} \sin \pi (\alpha + \beta + \gamma) \, \dot{\alpha}_{\beta} \, Q^{\beta}_{\gamma'}, \qquad \alpha + \beta + \gamma' \neq 1, 2, 3, \ldots \tag{3}$$

$$Q^{\alpha}_{\gamma'} = -\frac{e^{\mp \pi i \gamma'}}{\pi} \sin \pi (\alpha' + \beta' + \gamma) \, \dot{\alpha}_{\beta} \, Q^{\beta}_{\gamma'}, \qquad \alpha + \beta + \gamma \neq 1, 2, 3, \ldots \tag{4}$$

$$P^{\alpha} = e^{\mp \pi i \gamma} \, \frac{(m+1)_n}{(p+1)_n} \, P^{\beta}, \qquad \alpha' + \beta' + \gamma' = 1 + n, \; n = 0, 1, 2, \ldots \tag{5}$$

$$P^{\alpha} = e^{\mp \pi i \gamma'} \, \frac{(m+1)_n}{(p+1)_n} \, P^{\beta}, \qquad \alpha' + \beta' + \gamma = 1 + n, \; n = 0, 1, 2, \ldots \tag{6}$$

$$\beta_{\alpha'}\, p_{\gamma}^{\alpha'} = e^{\pm \pi i\,(\alpha'+\beta)}\, p^{\beta}, \qquad \alpha+\beta+\gamma = 1, 2, \ldots p \tag{7}$$

$$p^{\alpha} = e^{\pm \pi i\,(\alpha+\beta')}\, \alpha_{\beta'}\, p_{\gamma}^{\beta'}, \qquad \alpha+\beta+\gamma = 1, 2, \ldots m \tag{8}$$

$$p^{\alpha} = e^{\pm \pi i\,(\alpha+\beta')}\, \alpha_{\beta'}\, p_{\gamma'}^{\beta'}, \qquad \alpha+\beta+\gamma' = 1, 2, \ldots m \tag{9}$$

$$\left.\begin{aligned} & p_{\gamma}^{\alpha'} = e^{\mp \pi i \gamma} \frac{\Gamma(\alpha+\beta'+\gamma)\,\Gamma(m)}{\Gamma(\alpha'+\beta+\gamma)\,\Gamma(p)}\, p_{\gamma'}^{\beta'}, \\ & \text{provided that} \quad \alpha+\beta+\gamma = p+1,\ p+2, \ldots p+m \quad \text{if} \quad p \geqq m \\ & \qquad\qquad\qquad\quad \alpha+\beta+\gamma = m+1,\ m+2, \ldots m+p \quad \text{if} \quad p < m. \end{aligned}\right\} \tag{10}$$

Chapter IV

Confluent Hypergeometric Functions

§ 17. The solutions of KUMMER's differential equation

$$z\frac{d^2 y}{dz^2} + (\gamma - z)\frac{dy}{dz} - \alpha y = 0 \tag{1}$$

have been considered in recent papers by F. G. TRICOMI [38–40] and L. J. SLATER [33], to which we may refer the reader.

Using the familiar method of FROBENIUS we put

$$y = \sum_{s=0}^{\infty} c_s(\varrho)\, z^{\varrho+s}. \tag{2}$$

Substitution into the equation (1) yields the identity

$$\varrho(\varrho+\gamma-1)\, c_0(\varrho)\, z^{\varrho-1} + \sum_{s=0}^{\infty} \left[(\varrho+s+1)(\varrho+\gamma+s)\, c_{s+1}(\varrho) - (\varrho+\alpha+s)\, c_s(\varrho)\right] z^{\varrho+s} = 0.$$

If we determine the $c_s(\varrho)$'s such that

$$(\varrho+1+s)(\varrho+\gamma+s)\, c_{s+1}(\varrho) = (\varrho+\alpha+s)\, c_s(\varrho), \qquad s = 0, 1, 2, \ldots, \tag{3}$$

the series (2) will be a solution of the non-homogeneous equation

$$z\frac{d^2 y}{dz^2} + (\gamma - z)\frac{dy}{dz} - \alpha y = \varrho(\varrho+\gamma-1)\, c_0(\varrho)\, z^{\varrho-1}. \tag{4}$$

From (3) we get

$$c_s(\varrho) = \frac{(\varrho+\alpha)_s}{(\varrho+1)_s\,(\varrho+\gamma)_s}\, c_0(\varrho), \qquad s = 1, 2, 3, \ldots \tag{5}$$

By setting $\varrho = 0$ or $\varrho = 1 - \gamma$ and taking $c_0 = 1$ we see that (1) has the solutions

$$\sum_{s=0}^{\infty} \frac{(\alpha)_s z^s}{s!\,(\gamma)_s} = \Phi(\alpha, \gamma; z) \tag{6}$$

and

$$z^{1-\gamma}\,\Phi(\alpha - \gamma + 1, 2 - \gamma; z), \tag{7}$$

where $\Phi(\alpha, \gamma; z)$ is KUMMER's function. If γ is nonintegral, both of these solutions are applicable, and they are linearly independent. If $\gamma = 1$, the two solutions become identical, and if γ tends to an integer different from 1, one of them becomes meaningless.

1°. We suppose, first, that γ is an integer < 1 and that α is equal to one of the numbers $0, -1, -2, \ldots \gamma$. Setting $\varrho = 0$ the equations (3) leave c_0 and $c_{1-\gamma}$ indeterminate and (1) has the solution

$$c_0 \sum_{s=0}^{-\alpha} \frac{(\alpha)_s z^s}{s!\,(\gamma)_s} + c_{1-\gamma}\, z^{1-\gamma} \sum_{s=0}^{\infty} \frac{(\alpha - \gamma + 1)_s z^s}{s!\,(2-\gamma)_s},$$

where c_0 and $c_{1-\gamma}$ are arbitrary constants. For brevity's sake we put

$$\varphi(\alpha, \gamma; z) = \sum_{s=0}^{-\alpha} \frac{(\alpha)_s z^s}{s!\,(\gamma)_s}. \tag{8}$$

Besides (7) we then have the rational solution $\varphi(\alpha, \gamma; z)$.

2°. Next, we suppose that γ is an integer > 1, and that α is a positive integer $< \gamma$. Taking $\varrho = 1 - \gamma$ the equations (3) leave c_0 and $c_{\gamma-1}$ indeterminate. It follows that (1) has the solution

$$c_0\, z^{1-\gamma} \sum_{s=0}^{\gamma-\alpha-1} \frac{(\alpha - \gamma + 1)_s z^s}{s!\,(2-\gamma)_s} + c_{\gamma-1} \sum_{s=0}^{\infty} \frac{(\alpha)_s z^s}{s!\,(\gamma)_s},$$

containing the arbitrary constants c_0 and $c_{\gamma-1}$. Besides KUMMER's function (6) we then have the rational solution $z^{1-\gamma}\varphi(\alpha - \gamma + 1, 2 - \gamma; z)$.

3°. Now it is assumed that γ is a positive integer, and that $\alpha \neq 1, 2, 3, \ldots \gamma - 1$. Putting $\varrho = 1 - \gamma$, we can give $c_{\gamma-1}$ any value and we get again the solution (6). The two roots of the indicial equation then give the same solution. To get a second solution we observe that, if we take $c_{\gamma-1} = 1$, it follows from (3) that

$$c_{s+\gamma-1}(\varrho) = \frac{(\varrho + \alpha + \gamma - 1)_s}{(\varrho + \gamma)_s (\varrho + 2\gamma - 1)_s}, \qquad s \geq 1 - \gamma.$$

These functions are all regular at the point $\varrho = 1 - \gamma$. Particularly putting $s = 1 - \gamma$ we obtain

$$c_0(\varrho) = \frac{(\varrho + 1)(\varrho + 2)\ldots(\varrho + 2\gamma - 2)}{(\varrho + \alpha)(\varrho + \alpha + 1)\ldots(\varrho + \alpha + \gamma - 2)}.$$

It follows that $c_0(\varrho)$ has a zero of the first order at the point $\varrho = 1-\gamma$, and the right-hand side of (4) has consequently a zero of the second order at $\varrho = 1-\gamma$. The differential quotient of (2) is therefore a solution of (1), when $\varrho = 1-\gamma$. Hence, (1) has the solution

$$\left.\begin{aligned} G(\alpha, \gamma; z) &= \sum_{s=1}^{\gamma-1} (-1)^{s-1}(s-1)!\,\frac{(\alpha)_{-s}}{(\gamma)_{-s}} z^{-s} \\ &+ \sum_{s=0}^{\infty} \frac{(\alpha)_s z^s}{s!\,(\gamma)_s}\left([\alpha, \gamma; s] + \log z\right), \end{aligned}\right\} \quad (9)$$

where for $s = 1, 2, 3, \ldots$

$$[\alpha, \gamma; s] = \sum_{r=0}^{s-1}\left(\frac{1}{\alpha+r} - \frac{1}{\gamma+r} - \frac{1}{1+r}\right),$$

and

$$[\alpha, \gamma; 0] = 0.$$

Here the first term on the right-hand side of (9) should be replaced by zero if $\gamma = 1$. The series on the right of (9) converges for all finite values of z different from zero.

Another solution is obtained if c_0 is chosen in the following manner:

$$c_0(\varrho) = \frac{\Gamma(\varrho+\alpha)}{\Gamma(\varrho+1)\,\Gamma(\varrho+\gamma)}.$$

From (5) it now follows that

$$c_s(\varrho) = \frac{\Gamma(\varrho+\alpha+s)}{\Gamma(\varrho+1+s)\,\Gamma(\varrho+\gamma+s)},$$

and consequently (1) has the solution

$$\left.\begin{aligned} g(\alpha, \gamma; z) &= \sum_{s=1}^{\gamma-1} (-1)^{s-1}(s-1)!\,\frac{(\alpha)_{-s}}{(\gamma)_{-s}} z^{-s} + \\ &\sum_{s=0}^{\infty} \frac{(\alpha)_s z^s}{s!\,(\gamma)_s}\left[\Psi(\alpha+s) - \Psi(\gamma+s) - \Psi(1+s) + \log z\right]. \end{aligned}\right\} \quad (10)$$

If α tends to a non-positive integer, (10) becomes meaningless as $\Psi(\alpha)$ has poles at $\alpha = 0, -1, -2, \ldots$ To remedy this inconvenience we also consider the case

$$c_0(\varrho) = \frac{1}{\Gamma(1-\varrho-\alpha)\,\Gamma(\varrho+1)\,\Gamma(\varrho+\gamma)}$$

and thus get a solution denoted $g_1(\alpha, \gamma; z)$ and defined by

$$\left.\begin{aligned} g_1(\alpha, \gamma; z) &= \sum_{s=1}^{\gamma-1} (-1)^{s-1}(s-1)!\,\frac{(\alpha)_{-s}}{(\gamma)_{-s}} z^{-s} + \\ &\sum_{s=0}^{\infty} \frac{(\alpha)_s z^s}{s!\,(\gamma)_s}\left[\Psi(1-\alpha-s) - \Psi(\gamma+s) - \Psi(1+s) + \log(-z)\right], \end{aligned}\right\} \quad (11)$$

where α is not a positive integer. The series (10) and (11) converge for all finite values of z different from zero.

4°. If γ is a non-positive integer and α is different from $0, -1, -2, \ldots \gamma$, it is seen in the same manner that (1) has the solutions (7) and $z^{1-\gamma}\, G(\alpha-\gamma+1, 2-\gamma; z)$.

§ 18. From the expansions in powers of z it follows immediately that the solutions defined in § 17 are connected by the following linear relations

$$g(\alpha, \gamma; z) = G(\alpha, \gamma; z) + [\Psi(\alpha) - \Psi(\gamma) - \Psi(1)]\, \Phi(\alpha, \gamma; z), \tag{12}$$

$$g_1(\alpha, \gamma; z) = G(\alpha, \gamma; z) + [\Psi(1-\alpha) - \Psi(\gamma) - \Psi(1) \mp \pi i]\, \Phi(\alpha, \gamma; z), \tag{13}$$

$$g_1(\alpha, \gamma; z) = g(\alpha, \gamma; z) + \frac{\pi e^{\mp \pi i \alpha}}{\sin \pi\alpha}\, \Phi(\alpha, \gamma; z). \tag{14}$$

For KUMMER's function we have

$$\frac{d^n}{dz^n}\, \Phi(\alpha, \gamma; z) = \frac{(\alpha)_n}{(\gamma)_n}\, \Phi(\alpha+n, \gamma+n; z).$$

Also

$$\frac{d^n}{dz^n}\, g(\alpha, \gamma; z) = \frac{(\alpha)_n}{(\gamma)_n}\, g(\alpha+n, \gamma+n; z),$$

$$\frac{d^n}{dz^n}\, [z^{\gamma-1}\, g(\alpha, \gamma; z)] = (-1)^n (1-\gamma)_n\, z^{\gamma-n-1}\, g(\alpha, \gamma-n; z),$$

as is obvious by differentiation of the power series.

In KUMMER's equation (1) let $y = e^z y_1$ and $z = -z_1$. This transformation carries (1) into

$$z_1 \frac{d^2 y_1}{dz_1^2} + (\gamma - z_1) \frac{dy_1}{dz_1} - (\gamma - \alpha)\, y_1 = 0.$$

It is of the same form as (1). It follows that (1) has the solutions $\Phi(\alpha, \gamma; z)$ and $e^z\, \Phi(\gamma-\alpha, \gamma; -z)$ if $\gamma \neq 0, -1, -2, \ldots$, and it is easily seen that these two solutions are connected by KUMMER's relation

$$\Phi(\alpha, \gamma; z) = e^z\, \Phi(\gamma-\alpha, \gamma; -z), \tag{15}$$

a limiting case of EULER's relation

$$F(\alpha, \beta, \gamma; z) = (1-z)^{-\beta}\, F\left(\gamma-\alpha, \beta, \gamma; \frac{z}{z-1}\right)$$

for the Gaussian hypergeometric function. Furthermore, if γ is an integer > 1 and α is different from $1, 2, \ldots \gamma-1$, it follows that (1) has the solutions $\Phi(\alpha, \gamma; z)$, $G(\alpha, \gamma; z)$ and $e^z\, G(\gamma-\alpha, \gamma; -z)$. Among these three solutions there is a linear relation of the form

$$G(\alpha, \gamma; z) = C_1 e^z\, G(\gamma-\alpha, \gamma; -z) + C\, \Phi(\alpha, \gamma; z). \tag{16}$$

Multiply both sides by $z^{\gamma-1}$ and let $z \to 0$. We obtain $C_1 = 1$. Next, we expand e^z in powers of z and carry out the multiplication by G. If we equate the constant terms on both sides of (16), we obtain

$$C = \pm \pi i + \sum_{s=1}^{\gamma-1} \frac{1}{s} \frac{(1-\gamma)_s}{(1+\alpha-\gamma)_s},$$

which reduces to

$$\begin{aligned} C &= \pm \pi i + \Psi(\alpha-\gamma+1) - \Psi(\alpha) \\ &= \pm \pi i + \sum_{s=1}^{\gamma-1} \frac{1}{s-\alpha}. \end{aligned}$$

Thus (16) may be written

$$G(\alpha, \gamma; z) = e^z G(\gamma-\alpha, \gamma; -z) + \left(\pm \pi i + \sum_{s=1}^{\gamma-1} \frac{1}{s-\alpha} \right) \Phi(\alpha, \gamma; z), \tag{17}$$

the ambiguous sign being the same as the sign of $I(z)$. If $\gamma = 1$, (17) reduces to

$$G(\alpha, 1; z) = e^z G(1-\alpha, 1; -z) \pm \pi i \Phi(\alpha, 1; z). \tag{18}$$

If we equate the coefficients of z^n on both sides of (17), we obtain

$$\begin{aligned} \sum_{\nu=1}^{n} \frac{(-n)_\nu (\alpha)_\nu}{\nu! (\gamma)_\nu} [\alpha, \gamma; \nu] &= n! \sum_{\nu=1}^{\gamma-1} \frac{(1-\gamma)_\nu}{(1-\alpha)_\nu (\nu)_{n+1}} + \\ & \frac{(\gamma-\alpha)_n}{(\gamma)_n} \left([\gamma-\alpha, \gamma; n] + \sum_{s=1}^{\gamma-1} \frac{1}{\gamma-\alpha-s} \right). \end{aligned}$$

If $\gamma = 1$, this formula reduces to

$$\sum_{\nu=1}^{n} \frac{(-n)_\nu (\alpha)_\nu}{(\nu!)^2} [\alpha, 1; \nu] = \frac{(1-\alpha)_n}{n!} [1-\alpha, 1; n].$$

Eliminating G between the equations (12), (13), and (17), we find that

$$g(\alpha, \gamma; z) = e^z g_1(\gamma-\alpha, \gamma; -z), \tag{19}$$

where γ is a positive integer and α is not an integer $< \gamma$.
Similarly we derive the formula

$$g_1(\alpha, \gamma; z) = e^z g(\gamma-\alpha, \gamma; -z), \tag{20}$$

provided that γ is a positive integer and $\alpha \neq 1, 2, 3, \ldots$ Using KUMMER's relation (15) we can also write (19) and (20) in the following manner:

$$g(\alpha, \gamma; z) = e^z g(\gamma-\alpha, \gamma; -z) - \frac{\pi e^{\mp \pi i \alpha}}{\sin \pi\alpha} \Phi(\alpha, \gamma; z), \tag{21}$$

$$g_1(\alpha, \gamma; z) = e^z g_1(\gamma-\alpha, \gamma; -z) + \frac{\pi e^{\mp \pi i \alpha}}{\sin \pi\alpha} \Phi(\alpha, \gamma; z), \tag{22}$$

provided that α is not an integer. For the logarithmic solution $g(\alpha, \gamma; z)$ (19) is the analogue of Kummer's relation (15).

In the same manner we finally for the rational solution $\varphi(\alpha, \gamma; z)$ get the following relation[1]

$$\left.\begin{aligned} \varphi(\alpha, \gamma; z) &= e^z \varphi(\gamma-\alpha, \gamma; -z) \\ &- (-1)^{\alpha-\gamma} \frac{\Gamma(\alpha-\gamma+1)\Gamma(1-\alpha)}{\Gamma(2-\gamma)\Gamma(1-\gamma)} z^{1-\gamma} \Phi(\alpha-\gamma+1, 2-\gamma; z), \end{aligned}\right\} \quad (23)$$

if $\gamma = 0, -1, -2, \ldots$ and α is any of the numbers $0, -1, -2, \ldots \gamma$. When we replace γ by $2-\gamma$ and α by $\alpha-\gamma+1$, we obtain

$$\left.\begin{aligned} z^{1-\gamma} \varphi(\alpha-\gamma+1, 2-\gamma; z) &= e^z \varphi(1-\alpha, 2-\gamma; -z) \\ &+ (-1)^{\alpha} \frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)\Gamma(\gamma-1)} \Phi(\alpha, \gamma; z), \end{aligned}\right\} \quad (24)$$

provided that γ is an integer >1 and α is one of the numbers $1, 2, \ldots \gamma-1$.

§ 19. For Kummer's function we have the integral representation

$$\Phi(\alpha, \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 e^{zt} t^{\alpha-1} (1-t)^{\gamma-\alpha-1} dt,$$

provided that $\Re(\gamma) > \Re(\alpha) > 0$. This follows at once by expanding e^{zt} in powers of zt and integrating term by term. A further solution is Tricomi's function $\Psi(\alpha, \gamma; z)$ as defined by the Laplace integral

$$\Psi(\alpha, \gamma; z) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} e^{-zt} t^{\alpha-1} (1+t)^{\gamma-\alpha-1} dt, \quad (25)$$

if $\Re(\alpha) > 0$ and $\Re(z) > 0$. If we expand $(1+t)^{\gamma-\alpha-1}$ in powers of t, it is seen that this function admits the single asymptotic expansion[2]

$$\Psi(\alpha, \gamma; z) = \frac{1}{z^{\alpha}} \left[\sum_{\nu=0}^{n} \frac{(\alpha)_{\nu} (\alpha-\gamma+1)_{\nu}}{\nu! (-z)^{\nu}} + O(|z|^{-n-1}) \right] \quad (26)$$

valid in the sector $\frac{3\pi}{2} > \arg z > -\frac{3\pi}{2}$. From the Laplace-integral (25) we can easily deduce the Mellin-Barnes integral[3]

$$\Psi(\alpha, \gamma; z) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} z^{-t} \frac{\Gamma(t)\Gamma(t-\gamma+1)\Gamma(\alpha-t)}{\Gamma(\alpha)\Gamma(\alpha-\gamma+1)} dt, \quad (27)$$

[1] Cf. Watson, *Theory of Bessel Functions*, p. 103.
[2] See Erdélyi [7, p. 278].
[3] Cf. Erdélyi [7, p. 256].

provided that $\alpha \neq 0, -1, -2, \ldots;\ \alpha - \gamma \neq -1, -2, -3, \ldots$ and $|\arg z| < \frac{3\pi}{2}$. The path of integration is a line parallel to the imaginary axis, except that it is curved, if necessary, so that the decreasing sequences of poles lie to the left, and the increasing sequences of poles to the right of it. Let now $\Re(z) > 0$. Put $ze^{\pm\pi i}$ for z in (27) and multiply both sides by $e^{\pm\pi i\alpha}$. From the two equations thus obtained we get by subtraction

$$\Psi(\alpha, \gamma; z) = \frac{e^z}{2\pi i} \int_{\lambda - i\infty}^{\lambda + i\infty} z^{-t} \frac{\Gamma(t)\,\Gamma(t-\gamma+1)}{\Gamma(t+\alpha-\gamma+1)}\, dt, \tag{28}$$

where the poles lie to the left of the contour. The formula (28) has the advantage that there is no restriction upon the parameters. For the solution $e^z\,\Psi(\gamma-\alpha, \gamma; -z)$ we get from (28)

$$e^z\,\Psi(\gamma-\alpha, \gamma; -z) = \frac{1}{2\pi i} \int_{\lambda - i\infty}^{\lambda + i\infty} (-z)^{-t} \frac{\Gamma(t)\,\Gamma(t-\gamma+1)}{\Gamma(t-\alpha+1)}\, dt. \tag{29}$$

Evaluating the integrals on the right of (28) and (29) as $2\pi i$ times the sum of the residues at the poles, the following linear relations between the solutions are obtained (see Erdélyi [7, p. 259]):

$$\left.\begin{aligned} \Psi(\alpha, \gamma; z) &= \frac{\Gamma(1-\gamma)}{\Gamma(\alpha-\gamma+1)}\, \Phi(\alpha, \gamma; z) \\ &+ \frac{\Gamma(\gamma-1)}{\Gamma(\alpha)}\, z^{1-\gamma}\, \Phi(\alpha-\gamma+1, 2-\gamma; z), \quad \gamma \neq 0, \pm 1, \pm 2, \ldots \end{aligned}\right\} \tag{30}$$

$$\left.\begin{aligned} \Phi(\alpha, \gamma; z) &= e^{\pm\pi i\alpha} \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)}\, \Psi(\alpha, \gamma; z) \\ &+ e^{\pm\pi i(\alpha-\gamma)} \frac{\Gamma(\gamma)}{\Gamma(\alpha)}\, e^z\, \Psi(\gamma-\alpha, \gamma; -z), \quad \gamma \neq 0, -1, -2, \ldots \end{aligned}\right\} \tag{31}$$

$$\left.\begin{aligned} z^{1-\gamma}\, \Phi(\alpha-\gamma+1, 2-\gamma; z) &= -e^{\pm\pi i(\alpha-\gamma)} \frac{\Gamma(2-\gamma)}{\Gamma(1-\alpha)}\, \Psi(\alpha, \gamma; z) \\ &+ e^{\pm\pi i(\alpha-\gamma)} \frac{\Gamma(2-\gamma)}{\Gamma(\alpha-\gamma+1)}\, e^z\, \Psi(\gamma-\alpha, \gamma; -z), \quad \gamma \neq 2, 3, 4, \ldots \end{aligned}\right\} \tag{32}$$

$$\left.\begin{aligned} &g(\alpha, \gamma; z) = (-1)^\gamma\, \Gamma(\gamma)\, \Gamma(\alpha-\gamma+1)\, \Psi(\alpha, \gamma; z), \\ &\gamma = 1, 2, 3, \ldots \quad \alpha \neq \gamma-1, \gamma-2, \gamma-3, \ldots \end{aligned}\right\} \tag{33}$$

$$\left.\begin{aligned} &g_1(\alpha, \gamma; z) = (-1)^\gamma\, \Gamma(\gamma)\, \Gamma(1-\alpha)\, e^z\, \Psi(\gamma-\alpha, \gamma; -z), \\ &\gamma = 1, 2, 3, \ldots \quad \alpha \neq 1, 2, 3, \ldots \end{aligned}\right\} \tag{34}$$

$$\left.\begin{aligned} &z^{1-\gamma}\, g(\alpha-\gamma+1, 2-\gamma; z) = (-1)^\gamma\, \Gamma(\alpha)\, \Gamma(2-\gamma)\, \Psi(\alpha, \gamma; z), \\ &\gamma = 1, 0, -1, -2, \ldots \quad \alpha \neq 0, -1, -2, \ldots \end{aligned}\right\} \tag{35}$$

$$\left.\begin{array}{c} z^{1-\gamma} g_1(\alpha-\gamma+1, 2-\gamma; z) = -\Gamma(\gamma-\alpha)\Gamma(2-\gamma) e^z \Psi(\gamma-\alpha, \gamma; -z), \\ \gamma = 1, 0, -1, -2, \ldots \quad \alpha \neq \gamma, \gamma+1, \gamma+2, \ldots \end{array}\right\} \tag{36}$$

To these seven relations we shall add the following two

$$\left.\begin{array}{c} (-1)^\alpha (\gamma)_\alpha \varphi(\alpha, \gamma; z) = \Psi(\alpha, \gamma; z), \\ \gamma = 0, -1, -2, \ldots \quad \alpha = 0, -1, -2, \ldots \gamma, \end{array}\right\} \tag{37}$$

$$\left.\begin{array}{c} (\alpha)_{\gamma-\alpha-1} z^{1-\gamma} \varphi(\alpha-\gamma+1, 2-\gamma; z) = \Psi(\alpha, \gamma; z), \\ \gamma = 2, 3, 4, \ldots \quad \alpha = 1, 2, 3, \ldots \gamma-1. \end{array}\right\} \tag{38}$$

Substituting (26) in (33), we get the asymptotic expansion

$$g(\alpha, \gamma; z) \sim (-1)^\gamma \frac{\Gamma(\gamma)\Gamma(\alpha-\gamma+1)}{z^\alpha} \sum_{\nu=0}^{\infty} \frac{(\alpha)_\nu (\alpha-\gamma+1)_\nu}{\nu!\,(-z)^\nu}, \tag{39}$$

valid in the sector $\frac{3\pi}{2} > \arg z > -\frac{3\pi}{2}$. From (20) and (39) it now follows that

$$g_1(\alpha, \gamma; z) \sim (-1)^\gamma \frac{\Gamma(\gamma)\Gamma(1-\alpha)}{(-z)^{\gamma-\alpha}} e^z \sum_{\nu=0}^{\infty} \frac{(\gamma-\alpha)_\nu (1-\alpha)_\nu}{\nu!\, z^\nu}, \tag{40}$$

provided that $|\arg(-z)| < \frac{3\pi}{2}$. The asymptotic behavior of $g_1(\alpha, \gamma; z)$ is thus quite different from that of $g(\alpha, \gamma; z)$. From (31) we obtain for KUMMER's function the well-known asymptotic expansion

$$\left.\begin{array}{l} \Phi(\alpha, \gamma; z) \sim \dfrac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)} \displaystyle\sum_{\nu=0}^{\infty} \frac{(\alpha)_\nu (\alpha-\gamma+1)_\nu}{\nu!\,(-z)^{\alpha+\nu}} \\ \qquad + \dfrac{\Gamma(\gamma)}{\Gamma(\alpha)} e^z \displaystyle\sum_{\nu=0}^{\infty} \frac{(\gamma-\alpha)_\nu (1-\alpha)_\nu}{\nu!\, z^{\gamma-\alpha+\nu}}. \end{array}\right\} \tag{41}$$

Now use this result on the right of (12) and we get

$$G(\alpha, \gamma; z) \sim \frac{\Gamma(\gamma)}{\Gamma(\alpha)} (\Psi(\gamma) - \Psi(\alpha) + \Psi(1))\, e^z \sum_{\nu=0}^{\infty} \frac{(\gamma-\alpha)_\nu (1-\alpha)_\nu}{\nu!\, z^{\gamma-\alpha+\nu}}, \tag{42}$$

provided that $|\arg z| < \frac{\pi}{2}$. But if $\Re(z) \to -\infty$, so that $|\arg(-z)| < \frac{\pi}{2}$, we obtain the asymptotic expansion

$$G(\alpha, \gamma; z) \sim C \sum_{\nu=0}^{\infty} \frac{(\alpha)_\nu (\alpha-\gamma+1)_\nu}{\nu!\,(-z)^{\alpha+\nu}}, \tag{43}$$

where the constant C has the value

$$C = (-1)^{\gamma}\,\Gamma(\gamma)\,\Gamma(\alpha-\gamma+1)\left[e^{\mp\pi i\alpha} - \frac{\sin\pi\alpha}{\pi}\,(\Psi(\gamma)-\Psi(\alpha)+\Psi(1))\right],$$

that is

$$C = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)}\,[\Psi(\gamma)+\Psi(1)-\Psi(1-\alpha)\pm\pi i].$$

If $\alpha = \gamma$, (9) reduces to

$$G(\gamma,\gamma;z) = e^{z}\log z - \sum_{\nu=0}^{\infty}\frac{z^{\nu}}{\nu!}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{\nu}\right) - \sum_{\nu=1}^{\gamma-1}\frac{\Gamma(\nu)}{(-z)^{\nu}},$$

and for this solution we get from (42) and (43) the asymptotic expansion

$$G(\gamma,\gamma;z) \sim \Psi(1)\,e^{z} + \sum_{\nu=\gamma}^{\infty}\frac{\Gamma(\nu)}{(-z)^{\nu}}, \qquad |\arg z| \leq \pi.$$

If γ is an integer > 1 and α is one of the numbers $1, 2, \ldots \gamma-1$, the two series in (41) terminate and the asymptotic expansion (41) reduces to the relation (24), where the series are written in the reverse order.

References

[1]. P. Appel et J. Kampé de Fériet, *Fonctions hypergéométriques et hypersphériques, polynomes d'Hermite*. Paris 1926.

[2]. W. J. Archibald, The Complete Solution of the Differential Equation for the Confluent Hypergeometric Function. *Phil. Mag.* (7) 26 (1938), 415–419.

[3]. E. W. Barnes, Simple Types of Hypergeometric Series, *Trans. Camb. Phil. Soc.* 20 (1907), 253–270.

[4]. ——, A New Development of the Theory of the Hypergeometric Functions. *Proc. London Math. Soc.* (2) 6 (1908), 141–177.

[5]. N. G. de Bruijn, The Logarithmic Solutions of Linear Differential Equations. *Phil. Mag.* (7) 39 (1948), 134–140.

[6]. H. Buchholz, *Die konfluente hypergeometrische Funktion*, Berlin 1953.

[7]. A. Erdélyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, Higher Transcendental Functions. vol. 1, New York 1953.

[8]. A. Erdélyi and C. A. Swanson, Asymptotic Forms of Whittaker's Confluent Hypergeometric Functions. *Mem. Amer. Math. Soc.* 25 (1957).

[9]. W. L. Ferrar, On the Riemann P Equation with Integral Exponent Differences. *Proc. Edinburgh Math. Soc.* 43 (1925), 39–47.

[10]. C. F. Gauss, Disquisitiones generales circa seriem infinitam $1 + \frac{\alpha\beta}{\gamma} x + \cdots$ *Werke* 3, 123–162, Göttingen 1876.

[11]. ——, Determinatio seriei nostrae per aequationem differentialem secundi ordinis. *Werke* 3. 207–230.

[12]. E. Goursat, Sur l'équation différentielle linéaire qui admet pour intégrale la série hypergéométrique. *Ann. Sci. Ecole Norm. Sup.* (2) 10 (1881), 3–142.

[13]. ——, *Leçons sur les séries hypergéométriques*. Paris 1936.

[14]. M. J. Kampé de Fériet, *La fonction hypergéométrique*. Paris 1937.

[15]. A. Kienast, Untersuchungen über die Lösungen der Differentialgleichung $xy'' + (\gamma - x)\, y' - \beta y = 0$, *Denkschr. schweiz. naturf. Ges.* 57 (1921), 247–325.

[16]. F. Klein, *Vorlesungen über die hypergeometrische Function*, herausgegeben von O. Haupt. Berlin 1933.

[17]. A. Kratzer und W. Franz, *Transzendente Funktionen*. Leipzig 1960.

[18]. E. E. Kummer, Über die hypergeometrische Reihe. *J. reine angew. Math.* 15 (1836), 39–172.

[19]. A. H. Levelt, Hypergeometric Functions. *Indagationes Math.* 23 (1961), 361–403.

[20]. E. Lindelöf, Sur l'intégration de l'équation différentielle de Kummer. *Acta Soc. Scient. Fennicæ* 19 (1893), 3–31.

[21]. T. M. MacRobert, *Functions of a Complex Variable*. London 1954.

[22]. ——, Evaluation of an E-function when Two of the Upper Parameters Differ by an Integer. *Proc. Glasgow Math. Assoc.* 5 (1961), 30–34.

[23]. W. Magnus und F. Oberhettinger, *Formeln und Sätze für die speziellen Funktionen der mathematischen Physik*. Berlin 1948.

[24]. L. E. Mehlenbacher, The Interrelations of the Fundamental Solutions of the Hypergeometric Equation. Amer. J. Math. 60 (1938), 120–128.

[25]. Hj. Mellin, Über die fundamentale Wichtigkeit des Satzes von Cauchy für die Theorien der Gamma – und der hypergeometrischen Funktionen. *Acta. Soc. Scient.* Fennicæ 21 (1896).

[26]. ——, Abriss einer einheitlichen Theorie der Gamma – und der hypergeometrischen Funktionen. *Math. Ann* 68 (1910), 305–337.

[27]. A. Michaelsen, *Der logarithmische Grenzfall der hypergeometrischen Differentialgleichung n. Ordnung.* Kiel 1889.

[28]. N. E. Nørlund, Hypergeometric Functions. *Acta Math.* 94 (1955), 289–349.

[29]. F. W. J. Olver, The Asymptotic Solution of Linear Differential Equations of the Second Order in a Domain Containing one Transition Point. *Phil. Trans. Roy. Soc.* A 249 (1956), 65–97.

[30]. E. G. C. Poole, *Introduction to the Theory of Linear Differential Equations.* Oxford 1936.

[31]. B. Riemann, Beiträge zur Theorie der durch die Gauss'sche Reihe $F(\alpha, \beta, \gamma, x)$ darstellbaren Funktionen. *Ges. math. Werke*, Leipzig 1892, 67–87.

[32]. ——, Vorlesungen über die hypergeometrische Reihe. *G. m. W. Nachträge*, Leipzig 1902, 69–94.

[33]. L. J. Slater, *Confluent Hypergeometric Functions.* Cambridge 1960.

[34]. F. C. Smith, On the Logarithmic Solutions of the Generalized Hypergeometric Equation when $p = q + 1$. *Bull. Amer. Math. Soc.* 45 (1939), 629–636.

[35]. ——, Relations among the Fundamental Solutions of the Generalized Hypergeometric Equation. *Bull. Amer. Math. Soc.* 45 (1939), 927–935.

[36]. J. Tannery, Sur une équation différentielle linéaire du second ordre. *Ann. Sci. Ecole Norm. Sup.* (2) 8 (1879), 169–194.

[37]. L. W. Thomé, Zur Theorie der linearen Differentialgleichungen, *J. reine angew. Math.* 87 (1879), 222–349.

[38]. F. G. Tricomi, *Lezioni sulle Funzioni Ipergeometriche Confluenti.* Torino 1952.

[39]. ——, *Funzioni Ipergeometriche Confluenti*, Roma 1954.

[40]. ——, *Fonctions hypergéométriques confluentes*, Paris 1960.

[41]. G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge 1952.

[42]. E. T. Whittaker, On the Solution of Differential Equations by Definite Integrals, *Proc. Edinburgh Math. Soc.* (2) 2 (1931), 189–204.

[43]. E. Winkler, *Über die hypergeometrische Differentialgleichung n^{ter} Ordnung mit zwei endlichen singulären Punkten.* München 1931.

[44]. A. Winter, *Über die logarithmischen Grenzfälle der hypergeometrischen Differentialgleichungen.* Kiel 1905.

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